RESEARCH HIGHLIGHTS

(Last update: 3rd April 2013)

Here I briefly describe my contributions to research on numerical methods for hyperbolic balance laws that, in my view, have made an impact in the scientific community and have apparently survived the test of time.

WAF: Weighted Average Flux Method

The WAF numerical flux uses more wave information from the solution of the piece-wise constant Riemann problem than the Godunov method (first order), so as to obtain second-order accuracy in space and time, without data reconstruction. Ref [1] presents the idea for one-dimensional systems, while in Ref [2] the WAF method is extended to multidimensional problems on structured meshes.

[1] E F Toro. A Weighted Average Flux method for hyperbolic conservation laws. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences. Vol. 423, pages 401-418, 1989.

[2] S J Billett and E F Toro. On WAF-type schemes for multidimensional hyperbolic conservation laws. Journal of Computational Physics. Vol. 130, pages 1-24, 1997.

HLLC: Harten-Lax-van Leer Contact Riemann Solver

The Riemann solver of Harten Lax and van Leer (1983) assumes a two-wave structure, that is a two-wave model. This is the HLL solver. For systems with more than two equations HLL misses the intermediate characteristic fields leading to exceedingly diffusive numerical schemes. The idea of the HLLC scheme is to put back the missing intermediate characteristic fields. For the 3D Euler equations it is enough to restore just one linear field to account for the three fields corresponding to entropy and shear waves. The original idea was first presented in Ref [3], then published in [4] and [5]. There are numerous extensions and improvements to HLLC in the literature.

[3] E F Toro, M Spruce and W Speares. Restoration of the contact surface in the HLL Riemann solver. Technical report CoA 9204. Department of Aerospace Science, College of Aeronautics, Cranfield Institute of Technology. UK, June, 1992.

[4] E F Toro, M Spruce and W Speares. Restoration of the contact surface in the Harten-Lax-van Leer Riemann solver. Shock Waves. Vol. 4, pages 25-34, 1994.

[5] A Chakraborty and E F Toro. Development of an approximate Riemann solver for the steady supersonic Euler equations. The Aeronautical Journal. Vol. 98, pages 325-339, 1994.

FORCE: First Order Centred Flux

This is a centred numerical flux in which no explicit wave propagation information is used, that is the flux uses a 0-wave model. FORCE is derived from a re-interpretation of the staggered-grid version of Glimm's method, or Random Choice Method. The resulting conservative scheme is non-staggered and has CFL stability limit of unity. The original idea was first presented in Ref [6] and then included in [7]. Convergence is proved in Ref. [8] for two examples of non-linear systems of hyperbolic conservation laws. A variation of FORCE is PRICE: Primitive Centred Schemes. These schemes extend the idea of the FORCE flux to equations written in non-conservative form (no flux) and using the primitive or physical variables. The schemes are suitable for problems with smooth solutions or for systems with weak shocks. See [9]. A major advance is presented in [10], where FORCE is extended to multidimensional conservation laws solved on unstructured meshes. In Ref [11] FORCE is extended to non-conservative systems of hyperbolic balance laws.

[6] E F Toro. On Glimm-related schemes for conservation laws. Technical Report. Department of Mathematics and Physics. Manchester Metropolitan University, UK. 1996.

[7] E F Toro and S J Billett. Centred TVD schemes for hyperbolic conservation laws. IMA Journal of Numerical Analysis. Vol. 20, pages 47-79, 2000.

[8] G Q Chen and E F Toro. Centred schemes for non-linear hyperbolic equations. Journal of Hyperbolic Equations. Vol. 1, pages 531-566, 2004.

[9] E F Toro and A Siviglia. PRICE: primitive centred schemes for hyperbolic systems. International Journal for Numerical Methods in Fluids. Vol. 42, pages 1263-1291, 2003.

[10] E F Toro, A Hidalgo and M Dumbser. FORCE schemes on unstructured meshes I: conservative hyperbolic systems. Journal of Computational Physics. Vol. 228, pages 3368-3389, 2009.

[11] M Dumbser, A Hidalgo, M Castro, C Pares and E F Toro. FORCE schemes on unstructured meshes II. Non-conservative hyperbolic systems. Computer Methods in Applied Mechanics and Engineering. Vol. 199, Issues 9-12, pages 625-647, 2010.

MUSTA: Multistage Numerical Flux

The MUSTA flux is a centred flux, analogous to FORCE, which attempts to improve the accuracy in the representation of intermediate characteristic fields. This is achieved in a multi-stage fashion using a simple flux at each stage. No wave information is explicitly used. The original idea was first presented in [12] and then published in [13], [14] and [15]. The scheme has been found to be useful for very complex systems of equations, for which sufficiently simple and accurate Riemann solvers do no exist and might never be available.

[12] E F Toro. MUSTA: A multi-stage numerical flux. Isaac Newton Institute for Mathematical Sciences Preprint Series NI04008-NPA. University of Cambridge, UK. Available in pdf format at: http://www.newton.cam.ac.uk/preprints2004.

[13] E F Toro. A multi-stage numerical flux. Applied Numerical Mathematics. Vol. 56, pages 1464-1479, 2006.

[14] V A Titarev and E F Toro. MUSTA schemes for multi-dimensional hyperbolic systems: analysis and improvements. International Journal for Numerical Methods in Fluids. Vol. 49, pages 117-147, 2005.

[15] E F Toro and V A Titarev. MUSTA fluxes for systems of conservation laws. Journal of Computational Physics. Vol. 216, pages 403-429, 2006.

EVILIN: EVolved Initial condition LINearized Riemann solver.

The idea is this: given general initial, piece-wise constant data for the conventional Riemann problem I propose to evolve such data in a single step using a simple conservative method. In this manner large data becomes small data. Then, solve the Riemann for the evolved (small data) initial conditions, for which simple linearizations are justified and lead to close form solutions. The technique is easily applied to complicated hyperbolic systems. See Ref. [16].

[16]. E F Toro. Riemann solvers with evolved initial conditions. International Journal for Numerical Methods in Fluids. Vol. 52, pages 433-453, 2006.

ADER: Arbitrary Accuracy Derivative Riemann Problem Method

ADER represents a major step forward in the design of high-order non-linear numerical methods for hyperbolic balance laws. The schemes are also applicable to parabolic problems. The key idea is to define the local generalized Riemann problem (GRP), solve it and calculate a time integral average to obtain the numerical flux. The form of the scheme is identical to that of the first-order Godunov method, a one-step method. The GRP here is not that in which the initial condition is piece-wise linear (polynomials of degree one). The GRP here is the Cauchy problem in which (i) the initial condition is piece-wise smooth, such as polynomials of any degree for example, and (ii) if source terms are present in the equations, then these are included in the solution of the generalized Riemann problem. I also call this GRP, high-order Riemann problem.

The ADER schemes are fully discrete and use non-linear spatial reconstructions only once per time step. The ADER scheme is identical to the scheme of Harten and collaborators (1987) for the linear advection equation but differs for non-linear problems. The original ADER was presented in [17] for linear systems on Cartesian meshes. The scheme was then extended to non-linear one-dimensional problems in [18] and [19]. Major advances have been made by many collaborators (e.g Munz, Dumbser, Kaeser, Iske, Balsara and Castro) and other researchers. Current developments of the ADER methodology are set in both the finite volume and the discontinuous Galerkin finite element frameworks. See my complete list of publications. In such advances, it is not always easy to recognize the original ADER framework with its two components: non-linear reconstruction (ENO, WENO or other) followed by the solution of the Generalized Riemann problem to compute a numerical flux. The unified ADER framework is presented in [20], in which the various versions of ADER are simply due to the particular way of solving the generalized Riemann problem. For an introduction to ADER schemes see Chapters 19 and 20 of my book, Ref. [21].

The ADER schemes are beginning to make an impact on several areas of application, such as aero acoustics, seismic wave propagation, tsunami wave propagation and astrophysics.

[17] E F Toro, R C Millington and L A M Nejad. Towards very high-order Godunov schemes. In Godunov Methods: Theory and Applications. Edited Review. E F Toro (Editor). Kluwer Academic/Plenum Publishers. Conference in Honour of S K Godunov. Vol. 1, pages 897-902. New York, Boston and London, 2001.

[18] E F Toro and V A Titarev. Solution of the generalised Riemann problem for advection-reaction equations. Proceedings of the Royal Society of London. Series A. Vol. 458, pages 271-281, 2002.

[19] V A Titarev and E F Toro. ADER: arbitrary high order Godunov approach. Journal of Scientific Computing. Vol. 17, pages 609-618, 2002.

[20] G I Montecinos, C E Castro, M Dumbser and E F Toro. Comparison of solvers for the generalized Riemann problem for hyperbolic systems with source terms. Journal of Computational Physics. Vol. 231, pp 6472-6494, 2012.

[21] Toro E F. Riemann solvers and numerical methods for fluid dynamics. A practical introduction. 3rd Edition. Springer. Dordrecht, Heidelberg, London and New York, 2009.

DOT: Dumber-Osher-Toro Riemann Solver

A novel method to solve the classical Riemann problem is due to my young colleague Michael Dumbser. Effectively, DOT is a numerical generalization of the Osher-Solomon numerical flux. See Refs. [22], [23], [24]. Such generalization has turned out to be very powerful; it is applicable to any hyperbolic system.

[22] Michael Dumbser and Eleuterio Toro. A simple extension of the Osher Riemann solver to general non-conservative hyperbolic systems. Journal of Scientific Computing. Volume 48, pages 70-88, 2011.

[23] Michael Dumbser and Eleuterio Toro. On universal Osher-type schemes for general nonlinear hyperbolic conservation laws. Communications in Computational Physics. Vol. 10, pages 635-671, 2011.

[24] E F Toro and M Dumbser. Reformulated Osher-type Riemann solver. Computational Fluid Dynamics 2010. Springer-Verlag, Alexander Kuzmin (editor), 2011, pages 131-136.

NUMERICA: A Library of Source Codes for Teaching, Research and Applications. Free Software.

NUMERICA is a library of 50 source codes for solving hyperbolic partial differential equations using a broad range of modern, high resolution shock-capturing numerical methods. There are three sub-libraries: HYPER-LIN, HYPER-EUL and HYPER-WAT, see detailed references below, Refs. [25], [26] , [27] .The library is publicly available from my website.

[25] E F Toro. NUMERICA: A Library of Source Codes for Teaching, Research and Applications. HYPER-LIN. Methods for model hyperbolic equations. Numeritek Limited UK. ISBN 0-9536483-0-3. Software. 1999.

[26] E F Toro. NUMERICA: A Library of Source Codes for Teaching, Research and Applications. HYPER-EUL. Methods for the Euler equations. Numeritek Limited UK. ISBN 0-9536483-3-8. Software. 1999.

[27] E F Toro. NUMERICA: A Library of Source Codes for Teaching, Research and Applications. HYPER-WAT. Methods for the shallow water equations. Numeritek Limited UK. ISBN: 9536483-5-4. Software. 2000.