Godunov's Method for Blood Flow

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Contents:

- We study the Godunov method as applied to a non-linear hyperbolic system, and the blood flow equations in particular.

- We consider two approaches for computing the Godunov flux:

  - **State:** The first requires the calculation of *the Godunov state*, that is the state along the t-axis in the solution of the Riemann problem. Then, the numerical flux is simply the *physical flux* evaluated at this state.

  - **Flux:** In the second approach one calculates a numerical flux directly.

Note:
The contents of this chapter are based on the book:
E F Toro. *Riemann solvers and numerical methods for fluid dynamics*. Third Edition, Springer, 2009.

General initial-boundary value problem (IBVP)

- We apply the Godunov's method (Godunov, 1959) to a generic nonlinear hyperbolic system.
- Then we specialise to the blood flow equations
- Consider the initial-boundary value problem for any non-linear hyperbolic system

$$\left.\begin{aligned} &\text{PDEs:} && \partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \ , \quad x \in [a,b] \ , \quad t > 0 \ , \\ &\text{ICs:} && \mathbf{Q}(x,0) = \mathbf{Q}^{(0)}(x) \ , \quad x \in [a,b] \ , \\ &\text{BCs:} && \mathbf{Q}(a,t) = \mathbf{B}_L(t) \ , \quad \mathbf{Q}(b,t) = \mathbf{B}_R(t) \ , \quad t \geq 0 \ . \end{aligned}\right\} \tag{1}$$

- $\mathbf{Q}(x,t)$ is the vector of conserved variables; $\mathbf{F}(\mathbf{Q})$ is the flux function, or *physical flux*;
- $\mathbf{Q}^{(0)}(x)$ is the initial condition;
- $\mathbf{B}_L(t)$ and $\mathbf{B}_R(t)$ are the boundary conditions on the left and right boundaries respectively, two prescribed functions of time.

Finite volume method

Assuming IBVP (1) has a solution, the task at hand is to construct numerical methods to solve the problem approximately using the finite volume formula

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right) \tag{2}$$

with

$$\mathbf{Q}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{Q}(x, t_n)dx \tag{3}$$

and

$$\mathbf{F}_{i+\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{Q}(x_{i+\frac{1}{2}}, t))dt\ . \tag{4}$$

Formula (2) serves to update approximations to spatial integral averages (3) using numerical fluxes that are approximations to time integral averages (4) at the cell interface $x_{i+\frac{1}{2}}$. See Fig. 1.

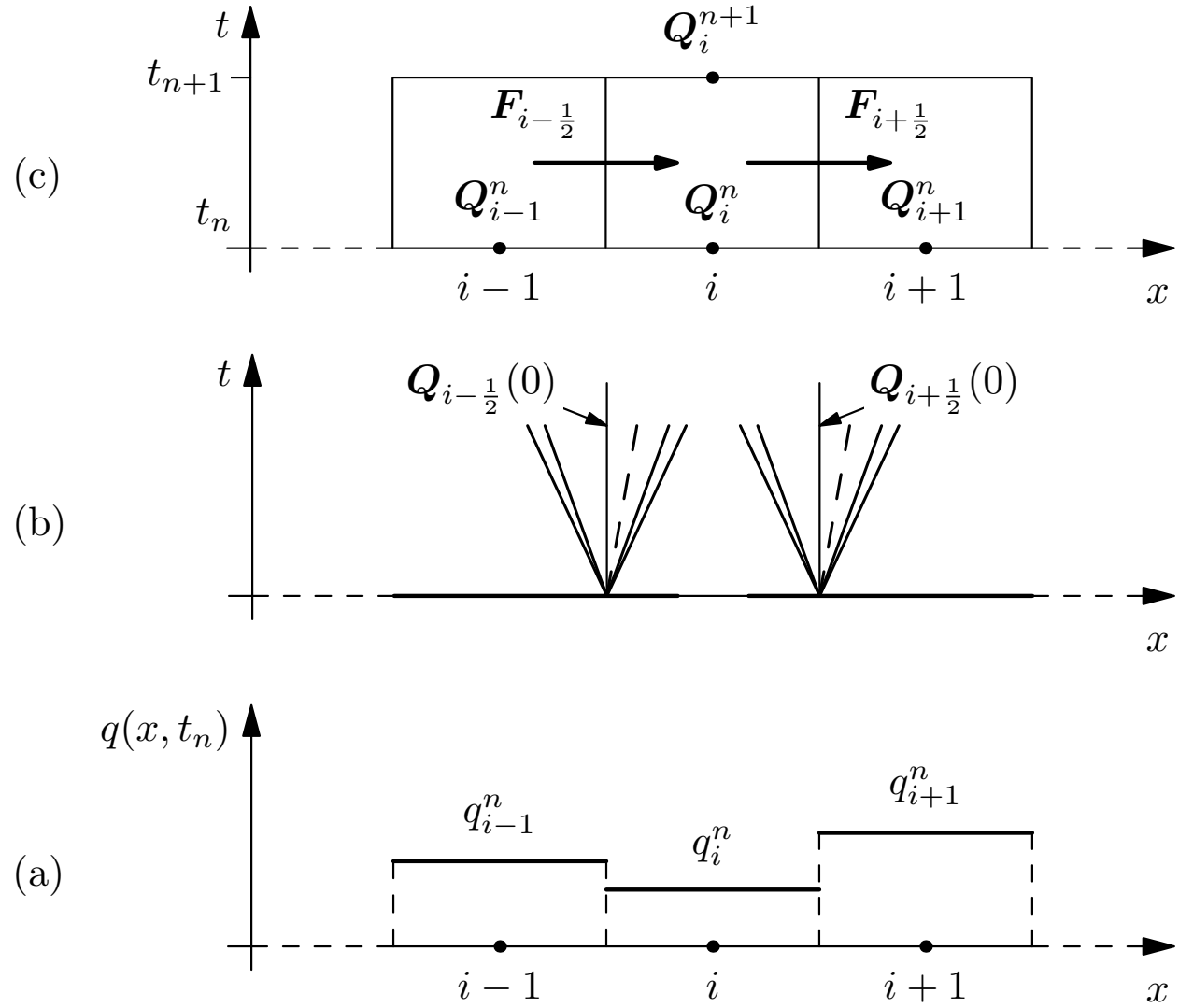

Fig. 1. Godunov's method for a hyperbolic system:
(a) integral averages give piece-wise constant data
(b) structure of solutions of Riemann problems at intercell boundaries
(c) finite volume formula to update averages using numerical fluxes.

Godunov flux

- To define the finite volume scheme (2) we prescribe suitable approximations to the integral (4) to obtain the *numerical flux* $\mathbf{F}_{i+\frac{1}{2}}$.
- The Godunov upwind numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ is computed from (4), making use of the solution $\mathbf{Q}_{i+\frac{1}{2}}(x/t)$ of the local Riemann problem

$$\left.\begin{array}{ll} \text{PDEs:} & \partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0}\ , \\ \\ \text{ICs:} & \mathbf{Q}(x,0) = \left\{ \begin{array}{lll} \mathbf{Q}_L \equiv \mathbf{Q}_i^n & \text{if} & x < 0\ , \\ \\ \mathbf{Q}_R \equiv \mathbf{Q}_{i+1}^n & \text{if} & x > 0\ . \end{array} \right. \end{array} \right\} \qquad (5)$$

- The Godunov flux [3] is computed from (4) and becomes

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{Q}_{i+\frac{1}{2}}(0))\ . \qquad (6)$$

- $\mathbf{Q}_{i+\frac{1}{2}}(0)$: *Godunov state*: $\mathbf{Q}_{i+\frac{1}{2}}(x/t)$ evaluated at interface $x/t = 0$.
- Note use of local coordinates.

Local coordinates

To deal with the local Riemann problems at each interface with changes lo local coordinates as follows:

$$\left.\begin{array}{lll} \bar{x} = x - x_{i+\frac{1}{2}} & , & \bar{t} = t - t^n \ , \\ x \in [x_i, x_{i+1}] & , & t \in [t^n, t^{n+1}] \ , \\ \bar{x} \in [-\frac{\Delta x}{2}, \frac{\Delta x}{2}] & , & \bar{t} \in [0, \Delta t] \ . \end{array}\right\} \tag{7}$$

We then use (x, t) to mean the local coordinates $(\hat{x}, \hat{t})$. See Fig. 2.

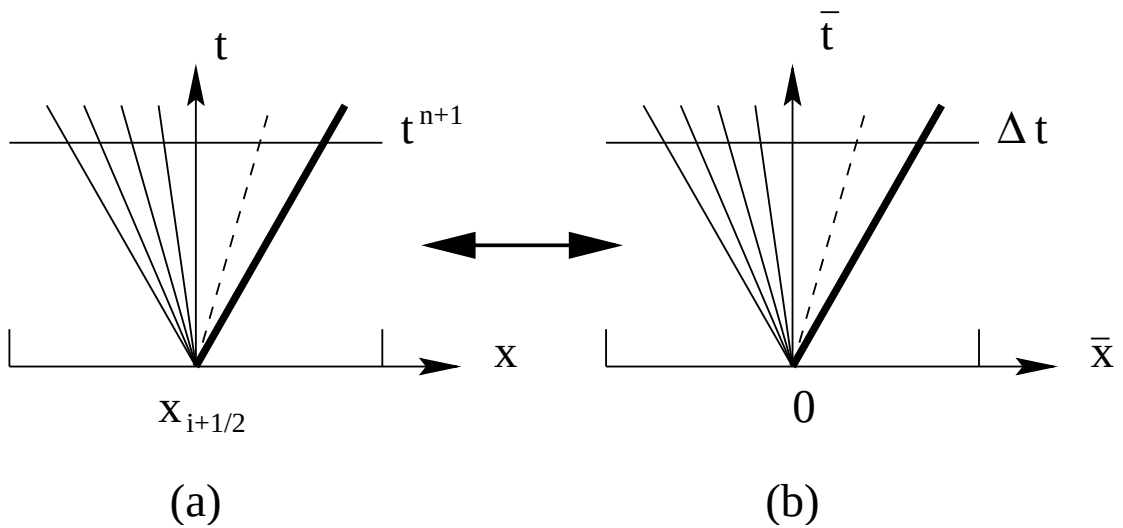

Figure 2. Correspondence between the global (a) and local (b) frames of reference for the solution of the Riemann problem.

Godunov flux with the exact Riemann solver

- **Star problem:** solution for A_* and u_* in the *Star Region*. One solves the non-linear equation (using Newton-Raphson method, for example)

$$f(A) \equiv f_L(A) + f_R(A) + \Delta u = 0 \ , \quad \Delta u \equiv u_R - u_L \ , \tag{8}$$

  where f_L and f_R are given in the lecture on the Riemann problem. Once the area $A = A_*$ has been found the velocity u_* is found as

$$u_* = \tfrac{1}{2}(u_L + u_R) + \tfrac{1}{2}[f_R(A_*) - f_L(A_*)] \ . \tag{9}$$

- **The sampling procedure** to find the Godunov state $\mathbf{Q}_{i+\frac{1}{2}}(0)$ for flux evaluation is applied with $\xi = x/t = 0$.
- **Passive scalar.** To choose the correct value of the passive scalar we use

$$\psi(x,t) = \begin{cases} \psi_L & \text{if} \quad \dfrac{x}{t} < u_* \ , \\ \\ \psi_R & \text{if} \quad \dfrac{x}{t} > u_* \ . \end{cases} \tag{10}$$

Approximate Riemann solvers: issues

- The Godunov method is the most accurate monotone (monotone for the scalar case) method
- For systems this is also true, but the accuracy depends crucially of the "Riemann solver"
- The exact solver is the best but with the shortcoming of (i) complexity and (ii) computational expense
- The computational expense is not excessive for blood flow, shallow water equations and ideal gas dynamics
- Approximate Riemann solvers can then be used but care is required in choosing the right approximation
- Approximate Riemann solvers are required to be:
  - **Complete:** their **wave model** contains all characteristic fields of the exact Riemann problem
  - **Non-linear.** Linearised Riemann solvers have various defects and are thus to be avoided
- The simplest Riemann solver is the Rusanov solver. Its wave model contains just one wave

Approximate Riemann solvers: a simple linearisation

- We look for approximations to A_* and u_* in the *Star Region*.
- First rewrite the governing equations in terms of primitive, or physical, variables A, u and ψ.

$$\partial_t \mathbf{P} + \mathbf{M}(\mathbf{P})\partial_x \mathbf{P} = \mathbf{0}\,, \tag{11}$$

with

$$\mathbf{P} = \begin{bmatrix} A \\ u \\ \psi \end{bmatrix}\,, \quad \mathbf{M}(\mathbf{P}) = \begin{bmatrix} u & A & 0 \\ c^2/A & u & 0 \\ 0 & 0 & u \end{bmatrix} \tag{12}$$

and initial conditions

$$\mathbf{P}_L = \begin{bmatrix} A_L \\ u_L \\ \psi_L \end{bmatrix}\,, \quad \mathbf{P}_R = \begin{bmatrix} A_R \\ u_R \\ \psi_R \end{bmatrix}\,. \tag{13}$$

- By assuming that $\mathbf{P}_L$ is close to $\mathbf{P}_R$ we linearise system (11) about mean values.
- The linear Riemann problem (12)-(13) is solved *exactly* by using standard methods. The solution in the *Star Region* is:

$$\left.\begin{aligned} A_* &= \frac{1}{2}(A_L + A_R) - \frac{1}{2}\frac{\tilde{A}}{\tilde{c}}(u_R - u_L)\ , \\ u_* &= \frac{1}{2}(u_L + u_R) - \frac{1}{2}\frac{\tilde{c}}{\tilde{A}}(A_R - A_L)\ . \end{aligned}\right\} \tag{14}$$

- The solution for $\psi(x, y)$ is as given by (10), though u_* is an approximation;
- Sampling procedure to find $\mathbf{Q}_{i+\frac{1}{2}}(0)$ for evaluating the numerical flux;
- This Riemann solver is very simple but not robust enough; fail for strong rarefactions, near the vacuum state;
- It would fail for trans-critical (or sonic) flow, leading to entropy violating solutions (or rarefaction shocks).
- This Riemann solver is **complete** but **linear**.

Sampling the solution: two options

- Option 1: insist on a linear structure for the solution and perform the sampling as for a linear system to find the Godunov state and thus the numerical flux

- Option 2: consider the **star values** as an approximation to the **Star Problem** in the exact solution. In this case we need to then perform a sampling of the solution as in the exact Riemann solver to find $\mathbf{Q}_{i+\frac{1}{2}}(0)$ and thus the numerical flux.

A two-rarefaction Riemann solver

- By assuming that both non-linear waves are rarefactions, constancy of Riemann invariants gives

$$u_* + 4c_* = u_L + 4c_L \ , \quad u_* - 4c_* = u_R - 4c_R \ , \tag{15}$$

- There follows

$$\left.\begin{aligned} u^*_{TR} &= \tfrac{1}{2}(u_L + u_R) - 2(c_R - c_L) \ , \\ c^*_{TR} &= \tfrac{1}{2}(c_L + c_R) - \tfrac{1}{8}(u_R - u_L) \ , \\ A^*_{TR} &= (\tfrac{2\rho}{\hat{\beta}})^2 (c^*_{TR})^4 \ . \end{aligned}\right\} \tag{16}$$

- The solution for $\psi(x, y)$ is as given by (10)
- Do not need to keep the assumption of two rarefactions, assume a fully non-linear structure
- Sample solution to find $\mathbf{Q}_{i+\frac{1}{2}}(0)$, as done in the exact Riemann problem
- This Riemann solver is **complete**, **non-linear** and very robust

The HLL Approximate Riemann Solver

- We want to solve the Riemann problem (43) approximately, with the aim of finding directly a numerical flux of the form

$$\mathbf{F}_0 = \frac{1}{T}\int_0^T \mathbf{F}(\mathbf{Q}(0,t))dt \tag{17}$$

  for an arbitrary time $T > 0$ and where $\mathbf{Q}(0,t)$ is an approximate solution of the Riemann problem along the t-axis.

- Here we construct a numerical flux following the HLL approach proposed by Harten, Lax and van Leer (1983) [4].

- We first establish some useful relations obtained by applying the integral form of the conservation laws applied to appropriately chosen control volumes.

Given initial conditions $\mathbf{Q}_L$ and $\mathbf{Q}_R$ for the Riemann problem, assume

- S_L is the speed of the fastest left travelling signals
- S_R is the speed of the fastest right travelling signals

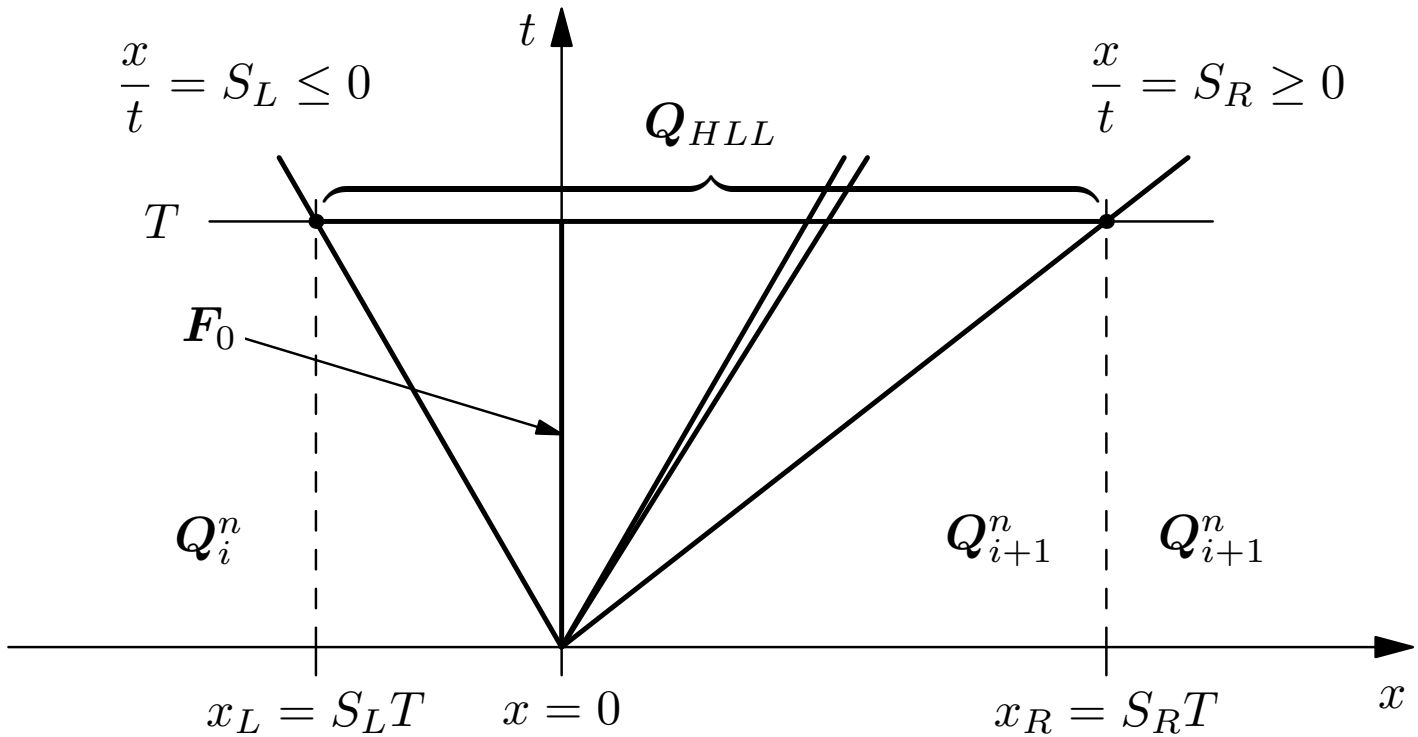

Fig. 3. Derivation of the HLL flux for a subcritical, or subsonic, wave pattern, $S_L \leq 0$ and $S_R \geq 0$.

- For an arbitrary time $T > 0$ we define the distances

$$x_L = TS_L \ , \quad x_R = TS_R \ . \tag{18}$$

- Consider the control volume $[x_L, 0] \times [0, T]$ in x-t space, see Fig. 3, and apply the integral form of the conservation laws

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \tag{19}$$

  in $[x_L, 0] \times [0, T]$.
- We obtain

$$\int_{x_L}^{0} \mathbf{Q}(x, T)dx = \int_{x_L}^{0} \mathbf{Q}(x, 0)dx + \int_{0}^{T} \mathbf{F}(\mathbf{Q}(x_L, t))dt - \int_{0}^{T} \mathbf{F}(\mathbf{Q}(0, t))dt \tag{20}$$

- Evaluation of the first and second terms on the right hand side gives

$$\int_{x_L}^{0} \mathbf{Q}(x, 0)dx = -S_L T \mathbf{Q}_L \ ; \quad \int_{0}^{T} \mathbf{F}(\mathbf{Q}(x_L, t))dt = T\mathbf{F}(\mathbf{Q}_L) \ . \tag{21}$$

Inserting these into (20) and dividing through by T gives

$$\mathbf{F}_0 = \frac{1}{T}\int_0^T \mathbf{F}(\mathbf{Q}(0,t))dt = -S_L\mathbf{Q}_L + \mathbf{F}(\mathbf{Q}_L) - \frac{1}{T}\int_{x_L}^0 \mathbf{Q}(x,T))dx \ . \quad (22)$$

- To define $\mathbf{F}_0$ approximately it is sufficient to find an approximation to the integral on the right hand side of (22).
- This is accomplished by finding an approximate state $\mathbf{Q}(x,T)$ by adopting an approach analogous to the two-step Lax-Wendroff method, or the Godunov centred method.
- Applying the integral form of the conservation laws (19) in the control volume $[x_L, x_R] \times [0, T]$, see Fig. 3, we obtain

$$\int_{x_L}^{x_R} \mathbf{Q}(x,T)dx = \int_{x_L}^{x_R} \mathbf{Q}(x,0)dx + \int_0^T \mathbf{F}(\mathbf{Q}(x_L,t))dt - \int_0^T \mathbf{F}(\mathbf{Q}(x_R,t))dt \ . \quad (23)$$

- The first term on the right hand side gives

$$\int_{x_L}^{x_R} \mathbf{Q}(x,0)dx = -S_L T\mathbf{Q}_L + S_R T\mathbf{Q}_R \ . \quad (24)$$

- Substitution of this expression into (23) and evaluating the remaining integrals in (23) yield

$$\int_{x_L}^{x_R} \mathbf{Q}(x,T)dx = T[S_R\mathbf{Q}_R - S_L\mathbf{Q}_L + \mathbf{F}(\mathbf{Q}_L) - \mathbf{F}(\mathbf{Q}_R)] \tag{25}$$

- On division through by $x_R - x_L = T(S_R - S_L)$ we obtain the averaged state

$$\left.\begin{aligned} \mathbf{Q}^{HLL} &= \frac{1}{(x_R - x_L)} \int_{x_L}^{x_R} \mathbf{Q}(x,T)dx \\ &= \frac{S_R\mathbf{Q}_R - S_L\mathbf{Q}_L + \mathbf{F}(\mathbf{Q}_L) - \mathbf{F}(\mathbf{Q}_R)}{S_R - S_L} . \end{aligned}\right\} \tag{26}$$

- We now use the state $\mathbf{Q}^{HLL}$ to evaluate the integral in (22). The resulting intercell flux is

$$\mathbf{F}_0 = \frac{S_R\mathbf{F}(\mathbf{Q}_L) - S_L\mathbf{F}(\mathbf{Q}_R) + S_LS_R(\mathbf{Q}_R - \mathbf{Q}_L)}{S_R - S_L} . \tag{27}$$

Then the HLL flux for the approximate Godunov method is

$$\mathbf{F}^{HLL}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & \text{if} \quad 0 \le S_L \ , \\ \dfrac{S_R\mathbf{F}(\mathbf{Q}_L) - S_L\mathbf{F}(\mathbf{Q}_R) + S_L S_R(\mathbf{Q}_R - \mathbf{Q}_L)}{S_R - S_L} \ , & \text{if} \quad S_L \le 0 \le S_R \ , \\ \mathbf{F}_R & \text{if} \quad 0 \ge S_R \ . \end{cases} \tag{28}$$

There are two issues to note regarding the HLL flux:

- **Wave speed estimates.** To apply the HLL scheme it is necessary to find estimates for S_L and S_R, for which variety of methods have been put forward
- **Missing intermediate waves.** The HLL approach takes an integral average of all the states lying between the fastest waves in the system. If the system consists of just two waves, then the **Two-Wave Model** of HLL is correct
- However if the system has more than two equations, then all intermediate waves get averaged. This results in numerical diffusion for intermediate waves
- **Restoring intermediate waves.** One can restore intermediate waves, as first suggested by Toro et al. (1994). See the HLLC solver.

Remarks on HLL

- HLL adopts a **two-wave model** for the hyperbolic system of interest, regardless of its dimension
- HLL is **complete** for a system of two equations, but **incomplete** for a larger system
- HLL applies to any hyperbolic system
- One needs wave speed estimates S_L and S_R, a non-trivial task, as knowledge of the exact solution of the Riemann problem is needed to perform this task correctly
- HLL is a **non-linear** Riemann solver and it is entropy satisfying
- Any **incomplete** Riemann solver has excessive numerical dissipation for intermediate waves, the missing waves in the model
- For selected systems HLL can be **completed** by following the **HLLC approach**

HLL and other Simple Methods

- **The Rusanov flux from HLL**
  - Define:

$$\left.\begin{array}{rcl} S^+ & = & max\{|S_L|, |S_R|\} \\ S_R & = & S^+ \,, \quad S_L = -S^+ \end{array}\right\} \tag{29}$$

  - Substitute these speeds into the HLL flux, to obtain the Rusanov flux

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}_L) + \mathbf{F}(\mathbf{Q}_R)\right] - \frac{1}{2}S^+(\mathbf{Q}_R - \mathbf{Q}_L) \tag{30}$$

  - The Rusanov scheme adopts a **one-wave model**. It is the simplest upwind, non-linear method, but incomplete for any system.
  - Sometimes the Rusanov flux is called the **Local Lax-Friedrichs Flux**, incorrectly in my view
- The (true) **Lax-Friedrichs Flux** can be obtained from Rusanov with $S^+ = \frac{\Delta x}{\Delta t}$

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}\left[\mathbf{F}(\mathbf{Q}_L) + \mathbf{F}(\mathbf{Q}_R)\right] - \frac{1}{2}\frac{\Delta x}{\Delta t}(\mathbf{Q}_R - \mathbf{Q}_L) \tag{31}$$

  This is the most diffusive of all stable schemes.

The HLLC Riemann Solver

The HLLC approximate Riemann solver (Toro et al. 1994) [7], [1] is a modification of the basic HLL scheme to account for the presence of intermediate waves in the structure of the solution of the Riemann problem. See Fig. 4.

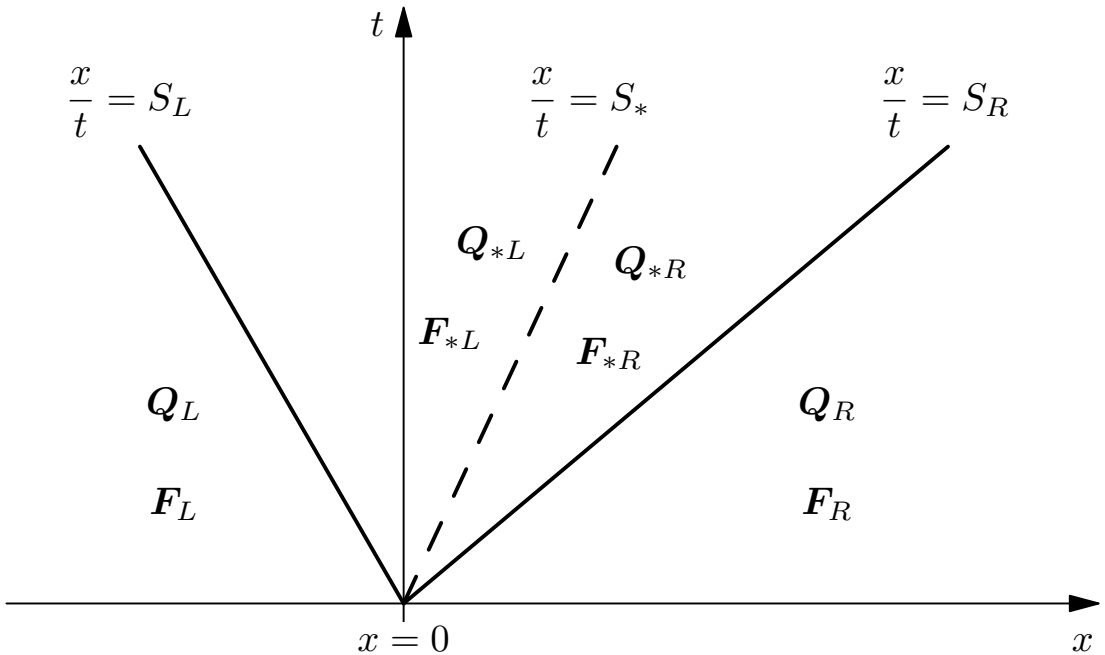

Fig. 4. Assumed wave pattern for the HLLC Riemann solver.

- Consider the wave pattern depicted in Fig. 4, where an intermediate wave of speed S_* is present.
- Application of the integral form of the conservation laws in $[x_L, 0] \times [0, T]$ and in $[0, x_R] \times [0, T]$ yield

$$\mathbf{F}_{*L} = \mathbf{F}_L + S_L(\mathbf{Q}_{*L} - \mathbf{Q}_L) \ , \quad \mathbf{F}_{*R} = \mathbf{F}_R + S_R(\mathbf{Q}_{*R} - \mathbf{Q}_R) \ . \tag{32}$$

  Here there are four unknown vectors $\mathbf{Q}_{*L}$, $\mathbf{Q}_{*R}$, $\mathbf{F}_{*L}$ and $\mathbf{R}_{*R}$.
- To solve the problem we make the following assumptions

$$A_{*L} = A_{*R} = A_* \ , \quad u_{*L} = u_{*R} = u_* = S_* \ . \tag{33}$$

- Assume in addition that an estimate $S_* = u_*$ is available. From the first component of the first vector equation in (32) we write

$$A_* u_* = A_L u_L + S_L(A_* - A_L) \tag{34}$$

- From the first component of the second vector equation in (32) we write

$$A_* u_* = A_R u_R + S_R(A_* - A_R) \ . \tag{35}$$

From (34) and (36) we write

$$A_* = \frac{A_R(u_R - S_R)}{u_* - S_R} = \frac{A_L(u_L - S_L)}{u_* - S_L} \tag{36}$$

From here we obtain

$$u_* = S_* = \frac{S_L A_R(u_R - S_R) - S_R A_L(u_L - S_L)}{A_R(u_R - S_R) - A_L(u_L - S_L)} \, . \tag{37}$$

If S_L and S_R are prescribed, then A_* and u_* are known from (36)-(37). Then the vectors $\mathbf{Q}_{*L}$ and $\mathbf{Q}_{*R}$ in (32) are given as

$$\mathbf{Q}_{*L} = A_* \begin{bmatrix} 1 \\ S_* \\ \psi_L \end{bmatrix} \, , \quad \mathbf{Q}_{*R} = A_* \begin{bmatrix} 1 \\ S_* \\ \psi_R \end{bmatrix} \, . \tag{38}$$

Now the vectors $\mathbf{F}_{*L}$ and $\mathbf{F}_{*R}$ in (32) are determined.

Finally, the HLLC flux is

$$\mathbf{F}^{HLLC}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & \text{if} \quad 0 \leq S_L \ , \\ \mathbf{F}_{*L} = \mathbf{F}_L + S_L(\mathbf{Q}_{*L} - \mathbf{Q}_L) & \text{if} \quad S_L \leq 0 \leq S_* \ , \\ \mathbf{F}_{*R} = \mathbf{F}_R + S_R(\mathbf{Q}_{*R} - \mathbf{Q}_R) & \text{if} \quad S_* \leq 0 \leq S_R \ , \\ \mathbf{F}_R & \text{if} \quad 0 \geq S_R \ , \end{cases} \tag{39}$$

where the states $\mathbf{Q}_{*L}$, $\mathbf{Q}_{*R}$ are given by (38).

- The use of HLLC instead of HLL for a system including the passive scalar ψ makes a dramatic difference to the resolution of the contact wave.
- This is particularly evident for long-time evolution.
- Pending problem: wave speed estimates

Wave Speed Estimates for HLLC

Assume an estimate for A^* using the TR approximation

$$A_{*TR} = (\frac{2\rho}{\hat{\beta}})^2 (c_{*TR})^4 \quad \text{with} \quad c_{*TR} = \frac{1}{2}(c_L + c_R) - \frac{1}{8}(u_R - u_L) \tag{40}$$

and compute

$$M_K = \sqrt{\frac{\gamma A_K A_{*TR}(A_{*TR}^{3/2} - A_K^{3/2})}{A_{*TR} - A_K}}, \quad K = L, R \tag{41}$$

Then evaluate the three wave speeds as follows:

$$S_L = \begin{cases} u_L - c_L & \text{if} \quad A_{*TR} \le A_L \quad (\textit{Left rarefaction}) \\ u_L - M_L/A_L & \text{if} \quad A_{*TR} > A_L \quad (\textit{Left shock}) \end{cases} \tag{42}$$

$$S_R = \begin{cases} u_R + c_R & \text{if} \quad A_{*TR} \le A_R \quad (\textit{Right rarefaction}) \\ u_R + M_R/A_R & \text{if} \quad A_{*TR} > A_R \quad (\textit{Right shock}) \end{cases} \tag{43}$$

$$S_* = \frac{S_L A_R(u_R - S_R) - S_R A_L(u_L - S_L)}{A_R(u_R - S_R) - A_L(u_L - S_L)} \tag{44}$$

Closing Remarks on Approximate Solvers

- The ideal Riemann solver is complete and non-linear
- The linearised Riemann solver is not an advisable choice, it can produce negative areas and rarefaction shocks (entropy fix needed)
- The TR solver is a recommended choice, but the sampling can be involved, as in the exact Riemann solver
- HLL is a very popular solver, but it is incomplete for a system of more than two equations.
- HLL for systems of more than 2 equations ignores intermediate characteristic fields leading to excessive numerical diffusion for these waves
- HLLC resolves the shortcomings of HLL, for our system of equations.

The Dumbser-Osher-Toro Riemann solver: DOT

We first give some background on the original Osher-Solomon approximate Riemann solver [5] and consider a general $m \times m$ hyperbolic system

$$\partial_t \mathbf{Q} + \partial_x \mathbf{F}(\mathbf{Q}) = \mathbf{0} \,, \tag{45}$$

with the vectors of conserved variables and fluxes respectively denoted as

$$\mathbf{Q} = [q_1, q_2, \ldots, q_m]^T \,, \quad \mathbf{F} = [f_1, f_2, \ldots, f_m]^T \,. \tag{46}$$

The real eigenvalues, written in increasing order, are denoted by $\lambda_i(\mathbf{Q})$ and the corresponding right eigenvectors by $\mathbf{R}_i(\mathbf{Q})$, for $i = 1, 2, \ldots, m$. Here we consider Godunov-type schemes of the form

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}) \,, \tag{47}$$

to solve (45), where $\mathbf{F}_{i+\frac{1}{2}}$ is the numerical flux found by solving the local Riemann problem for (45) with initial condition

$$\mathbf{Q}(x,0) = \begin{cases} \mathbf{Q}_0 & \text{if} \quad x < 0 \,, \\ \mathbf{Q}_1 & \text{if} \quad x > 0 \,. \end{cases} \tag{48}$$

Now recall that hyperbolicity of system (45) is equivalent to saying that the Jacobian matrix $\mathbf{A}(\mathbf{Q})$ of the flux $\mathbf{F}(\mathbf{Q})$ is diagonalizable, that is

$$\mathbf{A}(\mathbf{Q}) = \mathbf{R}(\mathbf{Q})\Lambda(\mathbf{Q})\mathbf{R}^{-1}(\mathbf{Q}) \,, \tag{49}$$

where $\mathbf{R}(\mathbf{Q})$ is the matrix formed by the right eigenvectors $\mathbf{R}_i(\mathbf{Q})$, $\mathbf{R}^{-1}(\mathbf{Q})$ is its inverse and $\Lambda(\mathbf{Q})$ is the diagonal matrix whose diagonal entries are the eigenvalues $\lambda_i(\mathbf{Q})$. We now introduce the definitions

$$\lambda_i^+(\mathbf{Q}) = \max(\lambda_i(\mathbf{Q}), 0) \,, \quad \lambda_i^-(\mathbf{Q}) = \min(\lambda_i(\mathbf{Q}), 0) \tag{50}$$

and consider the associated diagonal matrices $\Lambda^+(\mathbf{Q})$, $\Lambda^-(\mathbf{Q})$ and $|\Lambda^-(\mathbf{Q})|$, whose diagonal entries are $\lambda_i^+(\mathbf{Q})$, $\lambda_i^-(\mathbf{Q})$ and $|\lambda_i(\mathbf{Q})|$ respectively. Note that

$$|\lambda_i(\mathbf{Q})| = \lambda_i^+(\mathbf{Q}) - \lambda_i^-(\mathbf{Q}) \tag{51}$$

and hence

$$|\Lambda(\mathbf{Q})| = \Lambda^+(\mathbf{Q}) - \Lambda^-(\mathbf{Q}) \,. \tag{52}$$

Then we introduce

$$|\mathbf{A}(\mathbf{Q})| = \mathbf{R}(\mathbf{Q})|\Lambda(\mathbf{Q})|\mathbf{R}^{-1}(\mathbf{Q}) \, . \tag{53}$$

Osher and Solomon (1982) [5] defined the numerical flux as

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}(\mathbf{Q}_0) + \mathbf{F}(\mathbf{Q}_1)) - \frac{1}{2}\int_{\mathbf{Q}_0}^{\mathbf{Q}_1} |\mathbf{A}(\mathbf{Q})| d\mathbf{Q} \, . \tag{54}$$

- This requires the evaluation of an integral in phase space, which depends on the integration path chosen joining $\mathbf{Q}_0$ to $\mathbf{Q}_1$.
- Originally, Osher proposed two ways of choosing integration paths *so as to make the actual integration tractable*, called the P-ordering and the O-ordering.
- However, the analytical calculations are still too involved to be performed for general hyperbolic systems.
- Full details of the original Osher Riemann solver are found in Chapter 12 of Toro (2009) [6].

Dumbser and Toro (2011) [2] made two simple but effective suggestions:

- Choose any path, without constraints regarding the computational tractability of the scheme;
- Evaluate matrices by integrating numerically in phase space.

The simplest path is the *canonical path*

$$\psi(s;\mathbf{Q}_0,\mathbf{Q}_1) = \mathbf{Q}_0 + s(\mathbf{Q}_1 - \mathbf{Q}_0)\ , \quad s \in [0,1] \tag{55}$$

to evaluate the integral in (54).
Then, under a change of variables we obtain

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}(\mathbf{Q}_0) + \mathbf{F}(\mathbf{Q}_1)) - \frac{1}{2}\left(\int_0^1 |\mathbf{A}(\psi(s;\mathbf{Q}_0,\mathbf{Q}_1))|\, ds\right)(Q_1 - Q_0)\ . \tag{56}$$

Then the integral in (56) is computed *numerically* along the path ψ using a Gauss type quadrature rule with G points s_j and associated weights ω_j in the unit interval $I = [0,1]$.

We obtain

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}(\mathbf{Q}_0)+\mathbf{F}(\mathbf{Q}_1))-\frac{1}{2}\left(\sum_{j=1}^{G}\omega_j\,|\mathbf{A}(\psi(s_j;\mathbf{Q}_0,\mathbf{Q}_1))|\right)(Q_1-Q_0)\,. \tag{57}$$

Note that $|\mathbf{A}(\psi(s_j;\mathbf{Q}_0,\mathbf{Q}_1))|$ must be decomposed as in (53) for each s_j.

- The complete eigenstructure of the system is needed and is used at each integration point;
- The scheme is **non-linear and complete**, as it contains all characteristic fields of the exact problem.
- The scheme is very general. Osher's original version was restricted to very simple hyperbolic systems.
- It also applies to non-conservative hyperbolic systems.

Concluding remarks

- The Godunov method for the blood flow equations (for arteries) has been introduced. For veins it is necessary to use the appropriate tube law.
- The Godunov scheme works with the exact or approximate Riemann solvers.
- Apart from the exact Riemann solver, various approximate solvers have been introduced:
  - Simple linearised solver
  - Two rarefaction Riemann solver
  - HLL
  - HLLC
  - DOT
- These first-order Godunov schemes can be extended to high order of accuracy following a variety of procedures, such as: MUSCL-Hancock, WAF, ADER, etc.


A. Chakraborty and E. F. Toro.
Development of an Approximate Riemann Solver for the Steady Supersonic Euler Equations.
The Aeronautical Journal, 98:325–339, 1994.


M. Dumbser and E. F. Toro.
A simple extension of the Osher Riemann solver to non-conservative hyperbolic systems.
J. Sci. Comput., 48:70–88, 2010.


S. K. Godunov.
A Finite Difference Method for the Computation of Discontinuous Solutions of the Equations of Fluid Dynamics.
Mat. Sb., 47:357–393, 1959.


A. Harten, P. D. Lax, and B. van Leer.
On Upstream Differencing and Godunov–Type Schemes for Hyperbolic Conservation Laws.
SIAM Review, 25(1):35–61, 1983.


S. Osher and F. Solomon.

Upwind Difference Schemes for Hyperbolic Conservation Laws.
Math. Comp., 38,158:339–374, 1982.


E. F. Toro.
Riemann Solvers and Numerical Methods for Fluid Dynamics. A Practical Introduction. Third Edition.
Springer–Verlag, 2009.


E. F. Toro, M. Spruce, and W. Speares.
Restoration of the Contact Surface in the HLL–Riemann Solver.
Shock Waves, 4:25–34, 1994.