• We study some of the basic theory of non-linear hyperbolic equations
• We give three examples, the isothermal-gas equations, the non-linear shallow water equations and a non-linear model for blood flow in compliant vessels
First-order systems of equations

Here we study some very basic properties of nonlinear hyperbolic systems, starting from the general setting of $m$ hyperbolic balance laws in three space dimensions, written in differential conservative form

$$\partial_t Q + \partial_x F(Q) + \partial_y G(Q) + \partial_z H(Q) = S(Q),$$  \hspace{1cm} (1)

where

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix}; \quad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}; \quad G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}; \quad H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{bmatrix}; \quad S = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix}.\hspace{1cm} (2)$$

Independent variables: $x$, $y$, $z$ and $t$. $Q(x, y, z, t)$: vector of dependent variables, called conserved variables; $F(Q)$: flux vector in $x$-direction; $G(Q)$: flux vector in $y$-direction; $H(Q)$ flux vector in $z$-direction; $S(Q)$: vector of source terms.

Fluxes and sources are prescribed functions of $Q(x, y, z, t)$. 
Note that each component $k$ of the vectors in (2) is a function of the components $q_k = q_k(x, y, z, t)$ of the vector of unknowns $Q(x, y, z, t)$

$$
\begin{align*}
q_i &= q_i(x, y, z, t), \\
f_i &= f_i(q_1(x, y, z, t), \ldots, q_m(x, y, z, t)), \\
g_i &= g_i(q_1(x, y, z, t), \ldots, q_m(x, y, z, t)), \\
h_i &= h_i(q_1(x, y, z, t), \ldots, q_m(x, y, z, t)), \\
s_i &= s_i(x, y, z, t, q_1(x, y, z, t), \ldots, q_m(x, y, z, t)).
\end{align*}
$$

The model linear advection-reaction equation is an example of (1)

$$
\partial_t q(x,t) + \lambda \partial_x q(x,t) = \beta q(x,t), \quad \lambda : constant, \quad \beta : constant,
$$

with $f(q) = \lambda q$ and $s(q) = \beta q(x,t)$, both linear functions of $q(x, t)$.

An equation of the form (1) is called a system of balance laws. When $S(Q) = 0$ we call (1) a system of conservation laws (homogeneous equations); otherwise we speak of inhomogeneous equations.
One-dimensional systems

A general $m \times m$ one-dimensional non-linear system with source terms, written in differential conservation-law form reads

$$ \partial_t Q + \partial_x F(Q) = S(Q) \ . $$

(5)

$Q$: conserved variables, $F(Q)$: fluxes and $S(Q)$: sources. The integral form is:

$$ \int_{x_L}^{x_R} Q(x, t_2) \, dx = \int_{x_L}^{x_R} Q(x, t_1) \, dx$$

$$ + \int_{t_1}^{t_2} F(Q(x_L, t)) \, dt - \int_{t_1}^{t_2} F(Q(x_R, t)) \, dt$$

$$ + \int_{t_1}^{t_2} \int_{x_L}^{x_R} S(Q(x, t)) \, dx \, dt \ . $$

(6)
We note that an alternative integral form is

$$\frac{d}{dt} \int_{x_L}^{x_R} Q(x, t) \, dx = F(Q(x_L, t)) - F(Q(x_R, t)) + \int_{x_L}^{x_R} S(Q(x, t)) \, dx .$$

(7)

**Rankine-Hugoniot conditions.** Defining the jumps

$$\Delta F = F(Q_R) - F(Q_L) ; \quad \Delta Q = Q_R - Q_L$$

(8)

across a discontinuity of speed $S$, the Rankine-Hugoniot conditions for a system read

$$\Delta F = S \Delta Q .$$

(9)

Here $Q_L$ and $Q_R$ are the limiting states from the left and right of the discontinuity. Note that unlike the scalar case, now it is not possible to solve directly for the speed $S$ in terms of the jumps $\Delta F$ and $\Delta Q$. 
Quasi-linear form

It is convenient to express (5) in quasi-linear form

\[ \partial_t Q + A(Q)\partial_x Q = S(Q) , \]  

(10)

where \( A(Q) \) is the Jacobian matrix of \( F(Q) \), yet to be defined.

To illustrate the procedure we consider the special case

\[ \partial_t Q + \partial_x F(Q) = 0 , \]

(11)

\[ Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} , \quad F(Q) = \begin{bmatrix} f_1(q_1, q_2) \\ f_2(q_1, q_2) \end{bmatrix} . \]

Applying the chain rule to the flux term for each equation we have

\[ \partial_t q_1 + \frac{\partial}{\partial q_1} f_1(q_1, q_2) \partial_x q_1 + \frac{\partial}{\partial q_2} f_1(q_1, q_2) \partial_x q_2 = 0 , \]  

\[ \partial_t q_2 + \frac{\partial}{\partial q_1} f_2(q_1, q_2) \partial_x q_1 + \frac{\partial}{\partial q_2} f_2(q_1, q_2) \partial_x q_2 = 0 . \]  

(12)
In matrix form system (12) reads

\[
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
\partial_t
+ \begin{bmatrix}
\frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\
\frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2}
\end{bmatrix}
\partial_x
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
\] (13)

or

\[
\partial_t Q + A(Q)\partial_x Q = 0,
\] (14)

where

\[
A(Q) = \frac{\partial F}{\partial Q} =
\begin{bmatrix}
\frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} \\
\frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2}
\end{bmatrix}
\] (15)

is called the \textit{Jacobian matrix}. 
For an $m \times m$ one-dimensional non-linear, homogeneous, system (5) has quasi-linear form

$$\partial_t Q + A(Q) \partial_x Q = 0,$$  \hspace{1cm} (16)

with Jacobian matrix

$$A(Q) = \frac{\partial F}{\partial Q} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \ldots & \frac{\partial f_1}{\partial q_m} \\
\frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \ldots & \frac{\partial f_2}{\partial q_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \ldots & \frac{\partial f_m}{\partial q_m} \end{bmatrix}. \hspace{1cm} (17)$$

As done for linear systems we are now in a position to study the eigenstructure of first-order systems.
The concepts of eigenstructure and hyperbolicity for a non-linear system like are the same as for linear systems.

For non-linear systems the Jacobian, the eigenvalues and eigenvectors are functions $Q$.

The eigenvalues $A(Q)$ are the roots of the characteristic polynomial

$$P(\lambda) = | A(Q) - \lambda I | = 0 ,$$

where $I$ is the identity matrix and $\lambda$ is a parameter.

We assume the $m$ eigenvalues are written in increasing order

$$\lambda_1(Q) \leq \lambda_2(Q) \leq \ldots \leq \lambda_m(Q) .$$

We write eigenvectors in the order corresponding to their associated eigenvalues

$$R_1(Q) ; R_2(Q) ; \ldots ; R_m(Q)$$

$$L_1(Q) ; L_2(Q) ; \ldots ; L_m(Q) .$$
Hyperbolic system:

An $m \times m$ system (10) is **hyperbolic** if the Jacobian $A(Q)$ has $m$ real eigenvalues $\lambda_i(Q)$ ($i = 1, \ldots, m$) and a corresponding set of $m$ linearly independent eigenvectors $R_i(Q)$ ($i=1, \ldots, m$).

The system is said to be **strictly hyperbolic** if it is hyperbolic and all eigenvalues are distinct.

A three-dimensional $m \times m$ system (1) is hyperbolic if the matrix

$$D(Q) = \omega_1 A(Q) + \omega_2 B(Q) + \omega_3 C(Q)$$

(22)

has $m$ real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and a corresponding set of $m$ linearly independent eigenvectors $R_1, R_2, \ldots, R_m$, for all linear combinations (22), where the coefficients $\omega_1, \omega_2, \omega_3$ define a non-zero vector, that is

$$\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} > 0 .$$

(23)

$A, B, C$ are the Jacobian matrices associated to $F, G$ and $H$ in (1).
Nature of characteristic fields

For a hyperbolic system the characteristic speed $\lambda_i(Q)$ defines a characteristic field, the $\lambda_i$-field; we also speak of the $R_i$-field or simply the $i$-field.

Recall that the gradient of an eigenvalue $\lambda_i(Q)$ is given by

$$\nabla \lambda_i(Q) = \begin{bmatrix} \frac{\partial}{\partial q_1} \lambda_i, & \frac{\partial}{\partial q_2} \lambda_i, & \ldots, & \frac{\partial}{\partial q_m} \lambda_i \end{bmatrix}.$$  \hspace{1cm} (24)

A $\lambda_i$-characteristic field is said to be **linearly degenerate** if

$$\nabla \lambda_i(Q) \cdot R_i(Q) = 0, \quad \forall Q \in \mathbb{R}^m,$$  \hspace{1cm} (25)

where $\mathbb{R}^m$ is the set of real-valued vectors of $m$ components, called phase space, state space. For a $m \times m$ system we speak of phase plane. A $\lambda_i$-characteristic field is said to be **genuinely non-linear** if

$$\nabla \lambda_i(Q) \cdot R_i(Q) \neq 0, \quad \forall Q \in \mathbb{R}^m.$$  \hspace{1cm} (26)
Example 1: isentropic gas dynamics

\[ \partial_t Q + \partial_x F(Q) = 0 , \]

\[
Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \end{bmatrix} , \quad F(Q) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix} ,
\]

(27)

where \( \rho \) is density, \( u \) is velocity and \( p \) is pressure. These quantities are called the physical, or primitive, variables, as distinct from the conserved variables, which are the components of \( Q \). As there are two equations and three unknowns one requires a closure condition, called an equation of state, that expresses one variable in terms of the others. A simple closure is the isentropic equation of state that defines the pressure as

\[ p = C \rho^\gamma , \quad C = \text{constant} , \quad \gamma = \text{constant} . \]

(28)
To find the Jacobian we identify the conserved variables and express the flux components in terms of these.

\[ q_1 = \rho, \quad q_2 = \rho u, \quad f_1 = \rho u = q_2, \quad f_2 = \rho u^2 + p = \frac{q_2^2}{q_1} + Cq_1^\gamma. \quad (29) \]

Now we calculate the components of the Jacobian matrix. We have

\[
\begin{align*}
\frac{\partial}{\partial q_1} f_1(q_1, q_2) &= 0, & \frac{\partial}{\partial q_2} f_1(q_1, q_2) &= 1, \\
\frac{\partial}{\partial q_1} f_2(q_1, q_2) &= -\frac{q_2^2}{q_1^2} + a^2, & \frac{\partial}{\partial q_2} f_2(q_1, q_2) &= 2 \frac{q_2}{q_1},
\end{align*}
\]

(30)

\[ a^2 = \frac{d}{d\rho} p(\rho) = p'(\rho) = \gamma C q_1^{\gamma-1} = \frac{\gamma p}{\rho}. \quad (31) \]

Now the Jacobian in terms of \( u \) and the sound speed \( a = \sqrt{\frac{\gamma p}{\rho}} \) becomes

\[
A(Q) = \begin{bmatrix}
0 & 1 \\
-u^2 + a^2 & 2u
\end{bmatrix}. \quad (32)
\]
The eigenvalues of the system are the roots of the characteristic polynomial from (18)

\[ P(\lambda) = \lambda^2 - 2u\lambda + u^2 - a^2 = 0 , \]  
(33)

from which the eigenvalues are found to be

\[ \lambda_1 = u - a , \quad \lambda_2 = u + a . \]  
(34)

Exercise: Verify that the right and left eigenvectors are

\[ \mathbf{R} = \alpha_1 \begin{bmatrix} 1 \\ u - a \end{bmatrix} , \quad \mathbf{R}_2 = \alpha_2 \begin{bmatrix} 1 \\ u + a \end{bmatrix} . \]  
(35)

and

\[ \mathbf{L}_1 = \beta_1 \begin{bmatrix} 1 , \quad -(u + a) \end{bmatrix} , \quad \mathbf{L}_2 = \beta_2 \begin{bmatrix} -(u - a) , \quad 1 \end{bmatrix} . \]  
(36)

Here the coefficients \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are scaling factors.
Example 2: the nonlinear shallow water equations

The shallow water equations arise in the modelling of a wide variety of physical phenomena, such as water flows, atmospheric flows, dense gas dispersion, avalanches and even astrophysical flows. Here we study the augmented one-dimensional case with source terms

\[ \partial_t Q + \partial_x F(Q) = S(Q), \]  

(37)

with

\[
Q = \begin{bmatrix}
h \\
h u \\
h \psi
\end{bmatrix}, \quad F(Q) = \begin{bmatrix}
h u \\
h u^2 + \frac{1}{2}gh^2 \\
h u \psi
\end{bmatrix}, \quad S(Q) = \begin{bmatrix}
s_1 \\
s_2 \\
s_3
\end{bmatrix}. \tag{38}
\]

\(u(x,t)\): velocity, \(h(x,t)\): depth, \(\psi(x,t)\): a passive scalar, \(g\): acceleration due to gravity. Fig. 15 depicts the geometric situation. The depth \(h(x,t)\) is related to the free surface elevation \(H(x,t)\) and the bed elevation \(b(x) \geq 0\) above a horizontal datum via

\[ H(x,t) = b(x) + h(x,t). \]  

(39)
Fig. 15. Geometry of shallow water equations. The fixed bottom elevation is described by $b(x)$. The water depth is defined by $h(x, t)$, while the free surface position is given by $H(x, t) = b(x) + h(x, t)$. 
• The function $b(x)$ defines the bed profile and for the problems of interest here is prescribed and does not depend on time $t$;

• $g$ is the acceleration due to gravity taken as $g = 9.81 \, \text{m/s}^2$, a constant.

• There are two distinct situations of practical interest, namely the wet bed case in which the water depth $h$ is greater than zero and the dry bed case, in which portions of the bed are dry, that is $h = 0$.

• $S(Q)$ is a source term vector that accounts for various physical and geometric effects. For instance, when the bed elevation is variable then the source term vector becomes

$$S(Q) = \begin{bmatrix} 0 \\ -ghb'(x) \\ 0 \end{bmatrix}. \quad (40)$$

For many practical applications there will be additional terms in the vector $S(Q)$ to account for Coriolis forces, wind forces, bottom friction, etc.
Eigenstructure and characteristic fields

For the Jacobian we identify the conserved variables and write the flux as function of these.

\[ Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} h \\ hu \\ h\psi \end{bmatrix}, \quad F(Q) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} q_2 \\ q_2^2/q_1 + \frac{1}{2}gq_1^2 \\ q_2q_3/q_1 \end{bmatrix} \tag{41} \]

Simple calculations give all the entries of \( A(Q) \). For example

\[ \frac{\partial}{\partial q_1} f_1(q_1, q_2) = 0, \quad \frac{\partial}{\partial q_2} f_1(q_1, q_2) = 1, \quad \frac{\partial}{\partial q_3} f_1(q_1, q_2) = 0, \]

\[ \frac{\partial}{\partial q_1} f_2(q_1, q_2) = -\frac{q_2^2}{q_1^2} + gq_1 = a^2 - u^2, \]

where \( a \) is the celerity (analogous to the sound speed in gases) defined as

\[ a = \sqrt{gh}. \tag{42} \]
The Jacobian matrix is then given as

$$
A(Q) = \begin{bmatrix}
0 & 1 & 0 \\
\psi^2 - u^2 & 2u & 0 \\
-u\psi & \psi & u
\end{bmatrix}. \tag{43}
$$

**Eigenvalues.** The eigenvalues of $A$ are

$$
\lambda_1 = u - a, \quad \lambda_2 = u, \quad \lambda_3 = u + a, \tag{44}
$$

where $a$ is the celerity, as defined in (42). This is verified by finding the roots of the characteristic polynomial

$$
P(\lambda) = (u - \lambda) \left[ -\lambda(2u - \lambda) - (a^2 - u^2) \right] = 0, \tag{45}
$$

The eigenvalues are all real; they are also distinct under all circumstances, except for the case of dry bed $h = 0$, in which case $a = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = u$. 
The **right eigenvectors** are given by

\[
\mathbf{R}_1 = \alpha_1 \begin{bmatrix} 1 & u - a \\ \psi & 1 \end{bmatrix}, \quad \mathbf{R}_2 = \alpha_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{R}_3 = \alpha_3 \begin{bmatrix} 1 \\ u + a \end{bmatrix}
\] (46)

and the **left eigenvectors** are given by

\[
\mathbf{L}_1 = \beta_1 \begin{bmatrix} u + a, & -1, & 0 \end{bmatrix}, \quad \mathbf{L}_2 = \beta_2 \begin{bmatrix} -\psi, & 0, & 1 \end{bmatrix}, \quad \mathbf{L}_3 = \beta_3 \begin{bmatrix} u - a, & \end{bmatrix}
\] (47)

Here the coefficients \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \) and \( \beta_3 \) are **scaling factors**.

**Bi-orthonormality.** The reader can easily verify that the left and right eigenvectors (46), (47) are **bi-orthonormal**, that is they satisfy the relations

\[
\mathbf{L}_i \mathbf{R}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}
\] (48)

provided the scaling factors are chosen thus:

\[
\beta_1 = \frac{1}{2a\alpha_1}, \quad \beta_2 = \frac{1}{\alpha_2}, \quad \beta_3 = -\frac{1}{2a\alpha_3}.
\] (49)
Example 3: equations for blood flow

We consider here the case of constant material properties for blood vessels. The governing equations are

\[
\begin{align*}
\frac{\partial}{\partial t} A + \frac{\partial}{\partial x} (uA) &= 0, \\
\frac{\partial}{\partial t} (uA) + \frac{\partial}{\partial x} (\hat{\alpha} Au^2) + \frac{A}{\rho} \frac{\partial}{\partial x} p &= -Ru.
\end{align*}
\]

(50)

- $A(x, t)$ is cross-sectional area of vessel at position $x$ and time $t$
- $u(x, t)$ is averaged velocity of blood at a cross section
- $p(x, t)$ is pressure
- Blood density $\rho$ is assumed constant
- $R > 0$ is viscous resistance, a prescribed function
- $\hat{\alpha}$, coefficient that depends on assumed velocity profile
- There are 3 unknowns, 2 equations and thus a **tube law** is needed to close the system
Closure condition: the tube law

The **tube law** relates pressure $p(x, t)$ to wall displacement via $A(x, t)$. Here we adopt a very simple tube law of the form

$$p = p_{ext}(x, t) + \psi(A; \beta) \quad (51)$$

Here we adopt the simple tube law

$$\begin{align*}
\psi(A; \beta) &= \beta(x) \left( \sqrt{A} - \sqrt{A_0} \right), \\
\beta(x) &= \frac{\sqrt{\pi}}{(1 - \nu^2)} \frac{h_0(x)E(x)}{A_0(x)}. \\
\end{align*} \quad (52)$$

- $A_0(x)$ is the equilibrium cross-sectional area
- $h_0(x)$ is the vessel wall thickness
- $E(x)$ is the Young’s modulus
- $\nu$ is the Poisson ratio, taken to be $\nu = 1/2$
- $\psi(A; \beta) = p - p_{ext} \equiv p_{trans}$: transmural pressure
Simplified model: assumptions

Assume constant material properties:

- \( h_0 = constant \)
- \( A_0 = constant \)
- \( E = constant \)
- \( p_{ext} = constant \)
- \( R = 0 \)
- \( \hat{\alpha} = 1 \)

Therefore \( \beta(x) = \hat{\beta} \) in (80) is constant and the term \( \frac{A}{\rho} \partial_x p \) in (78) becomes

\[
\frac{A}{\rho} \partial_x p = \frac{\beta}{3\rho} \partial_x A^{3/2}. \tag{53}
\]

The equations now read

\[
\begin{aligned}
\partial_t A + \partial_x (uA) &= 0, \\
\partial_t (uA) + \partial_x (Au^2) + \frac{\hat{\beta}}{3\rho} \partial_x A^{3/2} &= 0.
\end{aligned} \tag{54}
\]
Now the equations can be written in conservation-law form

\[
\partial_t Q + \partial_x F(Q) = S(Q),
\]  

(55)

where

\[
Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \equiv \begin{bmatrix} A \\ Au \end{bmatrix}, \quad S(Q) = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ -Ru \end{bmatrix}
\]  

(56)

and the flux vector is

\[
F(Q) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} Au \\ Au^2 + \frac{\beta}{3\rho} A^{3/2} \end{bmatrix}.
\]  

(57)
The principal part of (55) written in quasi-linear form becomes

$$\partial_t Q + \frac{\partial F}{\partial Q} \partial_x Q = 0, \quad (58)$$

where

$$A(Q) = \frac{\partial F}{\partial Q} : \text{Jacobian matrix of the system} \quad (59)$$

To find $A(Q)$ one first expresses $F(Q)$ in terms of $Q$, namely

$$F(Q) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} Au \\ Au^2 + \frac{\beta}{3\rho} A^{3/2} \end{bmatrix} = \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + \frac{\beta}{3\rho} q_1^{3/2} \end{bmatrix} \quad (60)$$

$$A(Q) = \begin{bmatrix} \frac{\partial f_1(q_1, q_2)}{\partial q_1} & \frac{\partial f_1(q_1, q_2)}{\partial q_2} \\ \frac{\partial f_2(q_1, q_2)}{\partial q_1} & \frac{\partial f_2(q_1, q_2)}{\partial q_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\beta}{2\rho} \sqrt{A - u^2} & 2u \end{bmatrix}. \quad (61)$$
Proposition: The eigenvalues of (61) are all real and given by

\[ \lambda_1 = u - c, \quad \lambda_2 = u + c, \]  
where

\[ c = \sqrt{\frac{\hat{\beta} \sqrt{A}}{2\rho}} \]  
is the wave speed, analogous to the sound speed in gas dynamics.

Proof. By definition, the eigenvalues of system are the eigenvalues of the matrix \( A \), which in turn are the roots of the characteristic polynomial

\[ P(\lambda) = \text{Det}(A - \lambda I) = 0. \]  

\( I \) is the identity matrix and \( \lambda \) is a parameter. Simple calculations give

\[ P(\lambda) = \lambda^2 - 2u\lambda + u^2 - \frac{\hat{\beta} \sqrt{A}}{2\rho} = 0, \]  
from which the result (62) follows.
Right eigenvectors:

**Proposition:** The right eigenvectors of $A$ corresponding to the eigenvalues (62) are

$$
R_1 = \beta_1 \begin{bmatrix} 1 \\ u - c \end{bmatrix}, \quad R_2 = \beta_2 \begin{bmatrix} 1 \\ u + c \end{bmatrix},
$$

(65)

where $\beta_1$ and $\beta_2$ are arbitrary scaling factors.

**Proof.** For an arbitrary right eigenvector $R = [r_1, r_2]^T$ we have

$$
AR = \lambda R,
$$

(66)

which gives the algebraic system

$$
\begin{align*}
    r_2 &= \lambda r_1, \\
    (c^2 - u^2)r_1 + 2ur_2 &= \lambda r_2.
\end{align*}
$$

(67)

By substituting $\lambda$ in (67) by the appropriate eigenvalues in (62) in turn, we arrive at the sought result.
Formulation in terms of primitive variables

It can be easily shown that equations (82), in terms of the variables $A$ and $u$, may be written in quasi-linear form as

$$\partial_t V + M \partial_x V = S(V),$$

$$V = \begin{bmatrix} A \\ u \end{bmatrix}, \quad M = \begin{bmatrix} u & A \\ c^2/A & u \end{bmatrix}, \quad S(V) = \begin{bmatrix} 0 \\ -Ru/A \end{bmatrix},$$

with

$$c = \sqrt{\frac{\hat{\beta} \sqrt{A}}{2\rho}} : \text{wave speed}.$$  

This is accomplished by

- Expanding derivatives in the continuity equation (assuming smooth solutions)
- Expanding derivatives in the momentum equation and using the continuity equation

**Exercise:** Verify that (83)-(84) can be obtained from (82).
Concluding Remarks

- We have studied some basic mathematical properties of systems of non-linear hyperbolic equations.
- We have given examples of non-linear systems that have a physical meaning.
Exercises for non-linear hyperbolic systems
Problem 1: isentropic gas dynamics. Consider the isentropic equations of gas dynamics

\[
\partial_t Q + \partial_x F(Q) = 0, \\
Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad F(Q) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix}, \\
\] (71)

along with the *isentropic equation of state*

\[
p = C \rho^\gamma, \quad C = \text{constant}, \quad \gamma = \text{constant}. \quad (72)
\]

1. Find conditions on scaling parameters for the left and right eigenvectors to be orthonormal.
2. Determine the nature of the characteristic fields.

Problem 2: eigenstructure in terms of primitive variables. Consider the augmented shallow water equations on a non-horizontal channel

\[
\partial_t Q + \partial_x F(Q) = S(Q), \\
\] (73)
with

\[
Q = \begin{bmatrix}
  h \\
hu \\
h\psi
\end{bmatrix}, \quad
F(Q) = \begin{bmatrix}
  hu \\
hu^2 + \frac{1}{2}gh^2 \\
hu\psi
\end{bmatrix}, \quad
S = \begin{bmatrix}
  0 \\
  -ghb'(x) \\
  0
\end{bmatrix}.
\]

(74)

Here \( b(x) \) defines the channel bed above a horizontal datum and \( b'(x) \) is its gradient.

Assuming smooth solutions, by expanding the time and spatial partial derivatives show that the equations can be expressed in terms of the three variables \( h(x,t), u(x,t), \psi(x,t) \), called primitive variables, and that the governing equations are

\[
\begin{aligned}
  \partial_t h + u\partial_x h + h\partial_x u &= 0, \\
  \partial_t u + u\partial_x u + g\partial_x h &= -gb'(x), \\
  \partial_t \psi + u\partial_x \psi &= 0.
\end{aligned}
\]

(75)
Write these equations in matrix form

\[ \partial_t W + B \partial_x W = S(W) \]  \hspace{1cm} (76)

Find the eigenvalues and the left and right eigenvectors.

Is the system hyperbolic?

Determine the nature of the characteristic fields.

**Problem 3: eigenstructure in terms of conserved variables.**

1. Find the Jacobian matrix
2. Carry out the same tasks as in problem 2 but now in terms of the conserved variables, departing directly from (73).

**Problem 4: homogeneity property of a system.** A system of the form (73) is said to be *homogeneous* if the flux satisfies

\[ F(Q) = A(Q)Q, \]  \hspace{1cm} (77)

where \( A(Q) \) is the Jacobian matrix.

1. Verify that the shallow water equations do not satisfy the homogeneity property.
2 Find an approximate Jacobian matrix $\tilde{A}$ for the shallow water equations such that the homogeneity property is satisfied.

3 Check if the equations of isentropic gas dynamics satisfy the homogeneity property.

4 Verify that a linear hyperbolic system always satisfies the homogeneity property.

**Blood flow equations.** We consider the following one-dimensional, time-dependent non-linear mathematical model for blood flow

$$
\begin{align*}
\partial_t A + \partial_x (uA) &= 0, \\
\partial_t (uA) + \partial_x (\hat{\alpha}Au^2) + \frac{A}{\rho} \partial_x p &= -Ru,
\end{align*}
\tag{78}
$$

where $A(x, t)$ is cross-sectional area of vessel at position $x$ and time $t$; $u(x, t)$ is averaged velocity of blood at a cross section; $p(x, t)$ is pressure; $\rho$ is assumed constant; $R > 0$ is viscous resistance, a prescribed function; $\hat{\alpha}$, coefficient that depends on assumed velocity profile. Here we adopt a very simple tube law of the form

$$
p = p_{ext}(x, t) + \psi(A; K),
\tag{79}
$$
with

\[ \psi(A; K) = K(x) \left( \sqrt{A} - \sqrt{A_0} \right), \quad K(x) = \frac{\sqrt{\pi}}{(1 - \nu^2)} \frac{E(x)h_0(x)}{\sqrt{A_0(x)}}. \quad (80) \]

Here \( A_0(x) \) is the equilibrium cross-sectional area; \( h_0(x) \) is the vessel wall thickness; \( E(x) \) is the Young’s modulu; \( \nu \) is the Poisson ratio, taken to be \( \nu = 1/2 \). Assume \( h_0 = \text{constant} \); \( A_0 = \text{constant} \); \( E = \text{constant} \); \( p_{\text{ext}} = \text{constant} \); \( R = 0 \); \( \hat{\alpha} = 1 \).

**Problem 5**

1. Show that the term \( \frac{A}{\rho} \partial_x p \) in (78) becomes

\[ \frac{A}{\rho} \partial_x p = \frac{K}{3\rho} \partial_x A^{3/2}. \quad (81) \]

2. Show that the equations become

\[ \partial_t A + \partial_x (uA) = 0, \]

\[ \partial_t (uA) + \partial_x (Au^2) + \frac{K}{3\rho} \partial_x A^{3/2} = 0. \quad (82) \]
3. Write equations (82) in conservation-law form
4. Find the Jacobian matrix
5. Find (i) the eigenvalues, (ii) left and right eigenvectors
6. Is the system hyperbolic?
7. Determine the nature of the characteristic fields
8. Determine if the blood flow equations (78) are homogeneous.

**Problem 6: equations in terms of primitive variables.**

1. Show that equations (82), in terms of the variables $A$ and $u$, may be written in quasi-linear form as

$$\partial_t Q + A \partial_x Q = 0,$$

(83)

$$Q = \begin{bmatrix} A \\ u \end{bmatrix}, \quad A = \begin{bmatrix} u & A \\ \frac{c^2}{A} & u \end{bmatrix},$$

(84)

with

$$c = \sqrt{\frac{K\sqrt{A}}{2\rho}}: \text{ wave speed}.$$  

(85)
2. Find eigenvalues and left and right eigenvectors
3. Is the system hyperbolic?