The Riemann Problem for the Shallow Water Equations

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• We recall the 1D shallow water equations augmented by an extra equation for a passive scalar,

• We recall some mathematical properties of the equations,

• We solve exactly the Riemann problem.

Note:
The contents of this chapter are based on the book:
The Governing Equations

• The equation for conservation of mass reads

\[ \partial_t h + \partial_x (hu) = 0 , \]  

(1)

where \( h(x, t) \) is water depth and \( u(x, t) \) is the particle velocity.

• The equation for conservation of momentum reads

\[ \partial_t (hu) + \partial_x (hu^2 + \frac{1}{2}gh^2) = 0 , \]  

(2)

where \( g \) is the acceleration due to gravity.

• Recall that the celerity is defined as

\[ a = \sqrt{gh} , \]  

(3)

which is analogous to the speed of sound in a gas.
In certain applications it is of interest to consider an additional PDE

$$\partial_t \psi + u \partial_x \psi = 0.$$  \hfill (4)

$\psi(x, t)$ is transported with $u(x, t)$ and is often called a *passive scalar*. If we assume solutions are smooth, then from (1) and (4) we obtain a conservation equation

$$\partial_t (h \psi) + \partial_x (h \psi u) = 0.$$  \hfill (5)

Now the three equations of interest are (1), (2) and (5). These can be written in conservation form as

$$\partial_t Q + \partial_x F(Q) = 0,$$  \hfill (6)

with

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} h \\ hu \\ h\psi \end{bmatrix}, \quad F(Q) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \\ h\psi u \end{bmatrix}.$$  \hfill (7)
Quasi-linear Form and Eigenvalues

\[ \partial_t Q + A(Q) \partial_x Q = 0 , \]  

(8)

where \( A(Q) \) is the Jacobian matrix

\[ A(Q) = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \frac{\partial f_1}{\partial q_3} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \frac{\partial f_2}{\partial q_3} \\ \frac{\partial f_3}{\partial q_1} & \frac{\partial f_3}{\partial q_2} & \frac{\partial f_3}{\partial q_3} \end{bmatrix} . \]  

(9)

From (7) we have

\[ F(Q) = \begin{bmatrix} f_1(q_1, q_2, q_3) \\ f_2(q_1, q_2, q_3) \\ f_3(q_1, q_2, q_3) \end{bmatrix} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \\ h\psi u \end{bmatrix} = \begin{bmatrix} q_2 \\ q_1 + \frac{1}{2} gq_1^2 \\ q_2 q_3 \\ q_1 \end{bmatrix} . \]  

(10)
Calculating the partial derivatives in (9) and using $u$, $a$ and $\psi$ we have

$$A(Q) = \begin{bmatrix}
0 & 1 & 0 \\
-a^2 - u^2 & 2u & 0 \\
-u\psi & \psi & u
\end{bmatrix} .$$

(11)

The eigenvalues are the roots of the characteristic polynomial

$$P(\lambda) = Det(A - \lambda I) = 0 ,$$

(12)

where $I$ is the identity matrix and $\lambda$ is a parameter. It is easily verified that

$$P(\lambda) = (u - \lambda)[\lambda(2u - \lambda) + a^2 - u^2] = 0 ,$$

(13)

a cubic equation, for which three solutions exist, namely

$$\lambda_1 = u - a , \quad \lambda_2 = u , \quad \lambda_3 = u + a .$$

(14)

Note that all three roots are real; they are also distinct if $a \neq 0$. 
Right Eigenvectors

A right eigenvector $R$ corresponding to $\lambda$ satisfies

$$AR = \lambda R.$$  \hspace{1cm} (15)

For a generic $R = [r_1, r_2, r_3]^T$ we have

$$\begin{align*}
    r_2 &= \lambda r_1, \\
    (a^2 - u^2)r_1 + 2ur_2 &= \lambda r_2, \\
    -ur_1 + \psi r_2 + ur_3 &= \lambda r_3.
\end{align*}$$  \hspace{1cm} (16)

To find $R_i$ corresponding to $\lambda_i$ we substitute $\lambda_i$ into (16) and solve the resulting system for $r_1$, $r_2$ and $r_3$ in terms of a free parameter $\alpha_i$.

$$\begin{align*}
    R_1 &= \alpha_1 \begin{bmatrix} 1 \\ u - a \\ \psi \end{bmatrix}, & \quad R_2 &= \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, & \quad R_3 &= \alpha_3 \begin{bmatrix} 1 \\ u + a \\ \psi \end{bmatrix},
\end{align*}$$  \hspace{1cm} (17)

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are arbitrary scaling factors.
Left Eigenvectors

To compute a left eigenvector $L = [l_1, l_2, l_3]$ corresponding to an eigenvalue $\lambda$, we solve the system of algebraic equations

$$LA = \lambda L .$$

(18)

The left eigenvectors of $A$ are given by

$$L_1 = \beta_1 \left[ -(u + a) , \ 1 , \ 0 \right] ,$$

$$L_2 = \beta_2 \left[ -\psi , \ 0 , \ 1 \right] ,$$

$$L_3 = \beta_3 \left[ -(u - a) , \ 1 , \ 0 \right] ,$$

(19)

where the coefficients $\beta_1, \beta_2, \beta_3$ are arbitrary scaling factors.
The reader can easily verify that the right and left eigenvectors (17), (19) of the Jacobian matrix $A$ are bi-orthonormal, that is they satisfy the relations

$$L_i \cdot R_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

(20)

if the scaling factors are chosen thus

$$\beta_1 = \frac{1}{2a\alpha_1}, \quad \beta_2 = \frac{1}{\alpha_2}, \quad \beta_3 = -\frac{1}{2a\alpha_3}.$$ 

(21)
Nature of Characteristic Fields

First recall that a $\lambda_i$-characteristic field is said to be *linearly degenerate* if

\[
\nabla \lambda_i(Q) \cdot R_i(Q) = 0, \quad \forall Q \in \mathbb{R}^m
\]

(22)

\[
\nabla \lambda_i(Q) = \begin{bmatrix}
\frac{\partial}{\partial q_1} \lambda_i \\
\frac{\partial}{\partial q_2} \lambda_i \\
\vdots \\
\frac{\partial}{\partial q_m} \lambda_i
\end{bmatrix}^T.
\]

(23)

Now we show that the $\lambda_2$-characteristic field is linearly degenerate.

\[
\lambda_2(Q) = u = \frac{hu}{h} = \frac{q_2}{q_1}
\]

\[
\nabla \lambda_2(Q) = \begin{bmatrix}
\frac{\partial}{\partial q_1} \lambda_2 \\
\frac{\partial}{\partial q_2} \lambda_2 \\
\frac{\partial}{\partial q_3} \lambda_2
\end{bmatrix}^T = \begin{bmatrix}
-u \\
\frac{1}{h} \\
0
\end{bmatrix}^T.
\]

Then

\[
\nabla \lambda_2(Q) \cdot R_2(Q) = 0
\]

(24)

for $Q \in \mathbb{R}^3$ and thus the $\lambda_2$-characteristic field is *linearly degenerate*. 
The $\lambda_1$- and $\lambda_3$-characteristic fields are genuinely nonlinear. First recall that a $\lambda_i$-characteristic field is said to be genuinely non-linear if

$$\nabla \lambda_i(Q) \cdot R_i(Q) \neq 0, \ \forall Q \in \mathbb{R}^m.$$  (25)

Simple calculations give

$$\nabla \lambda_1(Q) \cdot R_1(Q) = -\frac{3}{2a} \neq 0 \quad \text{and} \quad \nabla \lambda_3(Q) \cdot R_3(Q) = \frac{3}{2a} \neq 0.$$  (26)

Therefore the $\lambda_1(Q)$- and $\lambda_3(Q)$-characteristic fields are genuinely non-linear, if $a \neq 0$. 
The Riemann Problem

The Riemann problem for the shallow water equations (6) is the initial value problem

\[
\begin{align*}
\text{PDEs:} & \quad \partial_t Q + \partial_x F(Q) = 0, \quad -\infty < x < \infty, \ t > 0, \\
\text{ICs:} & \quad Q(x, 0) = \begin{cases} 
Q_L & \text{if } x < 0, \\
Q_R & \text{if } x > 0.
\end{cases}
\end{align*}
\]  

(27)

- The vector of conservative variables \( Q \) and the vector of fluxes \( F(Q) \) are given in (7)
- \( Q_L \) and \( Q_R \) are two constant vectors that define the initial conditions of the problem.
Structure of the Solution of the Riemann Problem

\[
\frac{\partial Q}{\partial t} + \frac{\partial F(Q)}{\partial x} = 0
\]

\[
\lambda_3 = u + a
\]
\[
\lambda_2 = u
\]
\[
\lambda_1 = u - a
\]

**Fig. 1.** Structure of the solution of the Riemann problem for the augmented shallow water equations.

- There are three wave families separating four constant regions
- The outer waves are non-linear: shocks or rarefactions
- The middle wave is linear (called contact discontinuity)
- The solution in \( R_1 \) and \( R_2 \) (Star Region) is unknown
- Problem 1: *The Star Problem*
- Problem 2: *The Complete Solution*
Generalized Riemann Invariants (GRIs) are relations that apply across the wave structure of simple waves in $x - t$ space.

For a system of $m$ equations consider the $\lambda_j(Q)$-characteristic field and the corresponding right eigenvector

$$R_j = [r_{1j}, r_{2j}, \cdots, r_{mj}]^T .$$

(28)

The GRIs apply across the wave structure and lead to $m - 1$ ODEs in phase space:

$$\frac{dq_1}{r_{1j}} = \frac{dq_2}{r_{2j}} = \frac{dq_3}{r_{3j}} = \cdots = \frac{dq_m}{r_{mj}} .$$

(29)

The proof is omitted here.

Eqs. (29) relate ratios of $dq_i$ to $r_{ij}$ and we emphasize that the ratios are to be interpreted as meaning proportionality, that is

$$dq_i \propto r_{ij} .$$

If $r_{ij} = 0$ then $dq_i = 0$ and therefore $q_i$ does not change across the respective wave.
Assume a left rarefaction wave connecting $Q_L$ (left) and $Q_{*L}$ (right). See Fig. 5.

The rarefaction wave occupies a wedge $\mathcal{R}_L$ defined as

$$\mathcal{R}_L = \left\{ (x, t) / \; u_L - a_L \leq \frac{x}{t} \leq u_{*L} - a_{*L} \right\}.$$  \hspace{1cm} (30)

$\lambda_1(Q)$ increases monotonically across the wave from head to tail.
Application of GRIs across the $\lambda_1$-wave with $Q = [h, hu, h\psi]^T$ and $R_1 = [1, u - a, \psi]^T$ gives

$$\frac{dh}{1} = \frac{d(hu)}{u - a} = \frac{d(h\psi)}{\psi}.$$  \hspace{1cm} (31)

From first and third ratios $d\psi = 0$ and so across the $\lambda_1$ wave

$$\psi : \text{constant}.$$  \hspace{1cm} (32)

Analogously, from first and second ratios, along with integration in phase space we obtain

$$u + 2a = \text{constant}.$$  \hspace{1cm} (33)

From here we establish

$$u_*L + 2a_*L = u_L + 2a_L,$$  \hspace{1cm} (34)

which we also express as

$$u_*L = u_L - f_L; \quad f_L = 2(a_*L - a_L).$$  \hspace{1cm} (35)
**Solution inside rarefaction.** Consider a left rarefaction wave and a point inside the wave. See Fig. 6.

\[
\frac{x}{t} = \hat{u} - \hat{a}
\]

Fig. 6. Point \(\hat{P} = (\hat{x}, \hat{t})\) inside left rarefaction wave.
The point inside the rarefaction wave is \( \hat{P} = (\hat{x}, \hat{t}) \in \mathcal{R}_L \). Consider now a characteristic line through \( \hat{P} = (\hat{x}, \hat{t}) \) and the origin \((0, 0)\), of slope (known)

\[
\frac{\hat{x}}{\hat{t}} = \hat{u} - \hat{a} .
\] (36)

The unknowns of the problem are \( \hat{u} = u(\hat{x}, \hat{t}) \) and \( \hat{h} = h(\hat{x}, \hat{t}) \). Application of the left Riemann invariant (33) to connect the point \( \hat{P} \) to the left initial condition gives

\[
\hat{u} + 2\hat{a} = u_L + 2a_L .
\] (37)

Equations (36) and (37) are two equations for the two unknowns \( \hat{h} \) and \( \hat{u} \), whose solution is

\[
\hat{a}_L = a(\hat{x}, \hat{t}) = \frac{1}{3}(u_L + 2a_L - \frac{\hat{x}}{\hat{t}}) , \quad \hat{u}_L = u(\hat{x}, \hat{t}) = \frac{1}{3}(u_L + 2a_L + \frac{2\hat{x}}{\hat{t}}) .
\] (38)
Assume a right rarefaction wave, as depicted in Fig. 7, connecting the constant states $Q_{*R}$ (left) and $Q_R$ (right).

Fig. 7. Right rarefaction wave connecting states $Q_{*R}$ and $Q_R$. 

$$\frac{x}{t} = u_{*R} + a_{*R}$$

$$\frac{x}{t} = u_R + a_R$$
The wave occupies a wedge $\mathcal{R}_R$

$$\mathcal{R}_R = \left\{ (x, t)/ \ u_\ast R + a_\ast R \leq \frac{x}{t} \leq u_R + a_R \right\}. \quad (39)$$

$\lambda_3(Q)$ is monotone. The right generalized Riemann invariant gives

$$u - 2a = \text{constant}, \quad \psi: \text{constant}. \quad (40)$$

From here we obtain

$$u_\ast R - 2a_\ast R = u_R - 2a_R, \quad (41)$$

which we also express as

$$u_\ast R = u_R + f_R; \quad f_R = 2(a_\ast R - a_R). \quad (42)$$

The solution at $\hat{P} = (\hat{x}, \hat{t}) \in \mathcal{R}_R$ inside the right rarefaction wave is

$$\hat{a}_R = \frac{1}{3}(-u_R + 2a_R + \frac{\hat{x}}{\hat{t}}), \quad \hat{u}_R = \frac{1}{3}(u_R - 2a_R + \frac{2\hat{x}}{\hat{t}}). \quad (43)$$
Consider an isolated right-facing shock wave of speed $S_R$ associated with the $\lambda_3$-characteristic field, as depicted in Fig. 2.

Fig. 2. Right shock wave of speed $S_R$ connecting constant states $Q_R$ (ahead) and $Q_{*R}$ (behind).
\[ SR(Q_R - Q_{*R}) = F(Q_R) - F(Q_{*R}). \]  

(44)

In addition, the shock must also satisfy the Lax entropy condition

\[ \lambda_3(Q_{*R}) > SR > \lambda_3(Q_R). \]  

(45)

Characteristics run into the shock path, as illustrated in Fig. 2. Now we apply the transformation

\[ \hat{u}_{*R} = u_{*R} - SR, \quad \hat{u}_R = u_R - SR. \]  

(46)

Fig. 3. Right shock wave in transformed frame of reference.
In the new frame the shock propagation speed is 0 and the vectors of conserved variables and fluxes ahead of the shock are

\[
\hat{Q}_R = \begin{bmatrix}
    h_R \\
    h_R \hat{u}_R \\
    h_R \psi_R
\end{bmatrix},
\hat{F}_R = \begin{bmatrix}
    h_R \hat{u}_R \\
    h_R \hat{u}_R^2 + \frac{1}{2} gh_R^2 \\
    h_R \hat{u}_R \psi_R
\end{bmatrix},
\]

(47)

while those behind the shock are

\[
\hat{Q}_*R = \begin{bmatrix}
    h_*R \\
    h_*R \hat{u}_*R \\
    h_*R \psi_*R
\end{bmatrix},
\hat{F}_*R = \begin{bmatrix}
    h_*R \hat{u}_*R \\
    h_*R \hat{u}_*R^2 + \frac{1}{2} gh_*R^2 \\
    h_*R \hat{u}_*R \psi_*R
\end{bmatrix}.
\]

(48)

The Rankine-Hugoniot conditions in the moving frame are

\[
F(\hat{Q}_*R) - F(\hat{Q}_R) = 0 \times (\hat{Q}_*R - \hat{Q}_R),
\]

(49)

which give

\[
F(\hat{Q}_*R) = F(\hat{Q}_R).
\]
The above flux equality written in full gives

\[
\begin{align*}
    h_R \hat{u}_R &= h_R \hat{u}_R, \\
    h_R \hat{u}_R^2 + \frac{1}{2} g h_R^2 &= h_R \hat{u}_R^2 + \frac{1}{2} g h_R^2, \\
    h_R \hat{u}_R \psi_R &= h_R \hat{u}_R \psi_R.
\end{align*}
\]

(50)

The first equation in (50) says that the mass flux is constant across the shock,

\[- M_R \equiv h_R \hat{u}_R = h_R \hat{u}_R. \]

(51)

Using this into the third of equations (50) gives

\[\psi_R = \psi_R.\]

(52)

That is, \(\psi\) is constant across the shock wave. Thus we only need to work with the first two equations in (50); the second one gives

\[ (h_R \hat{u}_R) \hat{u}_R - (h_R \hat{u}_R) \hat{u}_R = \frac{1}{2} g (h_R^2 - h_R^2). \]

(53)

Use of (51) into (53) gives

\[ M_R = \frac{\frac{1}{2} g (h_R^2 - h_R^2)}{\hat{u}_R - \hat{u}_R}. \]

(54)
But from (51) we write

\[
\hat{u}_* R = -\frac{M_R}{h_* R} , \quad \hat{u}_R = -\frac{M_R}{h_R} .
\] (55)

Use of (55) into (54) followed by some manipulations yields

\[
M_R = \sqrt{\frac{1}{2} gh_R h_* R (h_R + h_* R)} .
\] (56)

From (46)

\[
u_* R = u_R + (\hat{u}_* R - \hat{u}_R) .
\] (57)

Inserting (55) into (57) followed by some algebraic manipulations gives

\[
u_* R = u_R + f_R ; \quad f_R = (h_* R - h_R) \sqrt{\frac{1}{2} g \frac{(h_* R + h_R)}{h_R h_* R}} .
\] (58)

From (46) we have

\[
S_R = u_R - \hat{u}_R .
\] (59)
Use of (55) into (59) followed by manipulations gives

\[ S_R = u_R + q_R a_R, \quad q_R = \sqrt{\frac{1}{2} \frac{(h_R + h_{*R}) h_{*R}}{h_{*R}^2}}. \]  \quad (60)

This expression relates the shock speed to the unknown depth \( h_{*R} \) behind the shock. Note that for the limiting case \( h_{*R}/h_R = 1 \) the shock speed coincides with the characteristic speed, that is \( S_R = u + a \), as expected.
For a left-facing shock of speed $S_L$ associated with the eigenvalue $\lambda_1 = u - a$ the analysis is similar to that of a right shock. See Fig. 4.

\[ \frac{dx}{dt} = S_L \]

Fig. 4. Left shock wave of speed $S_L$ connecting states $Q_L$ (ahead) and $Q_{*L}$ (behind).
First we define the transformation

\[ \hat{u}_* L = u_* L - S_L ; \quad \hat{u}_L = u_L - S_L . \]  \hspace{1cm} (61)

Then the Rankine-Hugoniot conditions give

\[
\begin{align*}
  h_* L \hat{u}_* L & = h_L \hat{u}_L , \\
  h_* L \hat{u}_*^2 L + \frac{1}{2} g h_*^2 L & = h_L \hat{u}_L^2 + \frac{1}{2} g h_L^2 , \\
  h_* L \hat{u}_* L \psi_* L & = h_L \hat{u}_L \psi_L .
\end{align*}
\] \hspace{1cm} (62)

The first of equations (62) says that the mass flux

\[ M_L \equiv h_* L \hat{u}_* L = h_L \hat{u}_L \] \hspace{1cm} (63)

is constant across the shock wave. Using this condition into the third of equations (62) gives

\[ \psi_* L = \psi_L . \] \hspace{1cm} (64)

In other words the passive scalar \( \psi \) is constant across the shock.
Analogous manipulations to those for a right-facing shock yield

\[ M_L = \sqrt{\frac{1}{2} gh_L h_{*L}(h_L + h_{*L})} \]  

(65)

and

\[ u_{*L} = u_L - f_L ; \quad f_L = (h_{*L} - h_L) \sqrt{\frac{1}{2} g \frac{(h_{*L} + h_L)}{h_L h_{*L}}} . \]  

(66)

This relates \( u_{*L} \) to \( h_{*L} \) via the function \( f_L \). Also, from (61)

\[ S_L = u_L - \hat{u}_L . \]  

(67)

Use of (63) into (67) followed by manipulations gives

\[ S_L = u_L - q_L a_L ; \quad q_L = \sqrt{\frac{1}{2} \frac{(h_L + h_{*L}) h_{*L}}{h_L^2}} . \]  

(68)

This expression relates \( S_L \) to \( h_{*L} \). Again, in the limiting case \( h_{*L}/h_L = 1 \) we have \( S_L = u - a \).
An isolated contact discontinuity connecting the (constant) states $Q^*_L$ and $Q^*_R$ associated with the $\lambda_2$-characteristic field is depicted in Fig. 8.

Fig. 8. Contact wave (associated with the linearly degenerate field $\lambda_2$) connecting states $Q^*_L$ and $Q^*_R$. 

The wave is a single discontinuity travelling with speed $u_*$ and characteristics either side of the discontinuity run parallel to it, namely

$$\lambda_2(Q_{*L}) = u_* = \lambda_2(Q_{*R}). \quad (69)$$

An eigenvector analysis provides the sought jump conditions across the contact discontinuity. The right eigenvector corresponding to $\lambda_2$ is $R_2 = [0, 0, 1]^T$, from which we have

$$\begin{align*}
   u_{*L} &= u_{*R} = u_* , \\
   h_{*L} &= h_{*R} = h_* , \\
   \psi_{*L} &\neq \psi_{*R}.
\end{align*} \quad (70)$$

**Exercise.** Show that the above solution satisfies the Rankine-Hugoniot conditions across the contact discontinuity.
Fig. 9 depicts the Riemann problem. The left and right waves can be shocks or rarefactions. The velocity and depth are constant in the Star Region; $\psi$ is also constant in $\mathcal{R}_1$ and $\mathcal{R}_2$ but with a discontinuous jump across the contact wave.

![Diagram of the solution of the Riemann Problem for the augmented shallow water equations.](image)

Fig. 9. Structure of the solution of the Riemann Problem for the augmented shallow water equations.
To find the velocity \( u_\ast \) and the depth \( h_\ast \)

- We assemble together all the wave relations derived for each elementary wave in isolation.
- Velocity \( u_\ast \) is connected to \( Q_L \) via a function \( f_L \) and
- Velocity \( u_\ast \) is connected to \( Q_R \) via a function \( f_R \).
- \( f_L \) and \( f_R \) depend on the unknown depth \( h_\ast \), the wave type (shock or rarefaction) and, parametrically, on the initial conditions \( Q_L \) and \( Q_R \), that is

\[
f_L = f_L(h_\ast, w_L; Q_L) ; \quad f_R = f_R(h_\ast, w_R; Q_R) . \tag{71}
\]

Here \( w_L \) and \( w_R \) denote logical variables that identify the wave type; \( w_K \) denotes either a shock or a rarefaction, for \( K = L \) and \( K = R \).
- The complete solution procedure for the Star Problem is then summarised in the following proposition.
Proposition: The solution $h_*$ for the Riemann problem (27) is the root of

$$f(h) \equiv f_L(h, w_L; h_L) + f_R(h, w_R; h_R) + \Delta u = 0 \ , \ \Delta u \equiv u_R - u_L \ , \ (72)$$

$$f_L(h, w_L; h_L) = \begin{cases} 
2(\sqrt{gh} - \sqrt{gh_L}) & \text{if } h \leq h_L \quad (w_L: \text{rarefaction}), \\
(h - h_L) \sqrt{\frac{1}{2} g \frac{(h + h_L)}{hh_L}} & \text{if } h > h_L \quad (w_L: \text{shock}), 
\end{cases} \quad (73)$$

$$f_R(h, w_R; h_R) = \begin{cases} 
2(\sqrt{gh} - \sqrt{gh_R}) & \text{if } h \leq h_R \quad (w_R: \text{rarefaction}), \\
(h - h_R) \sqrt{\frac{1}{2} g \frac{(h + h_R)}{hh_R}} & \text{if } h > h_R \quad (w_R: \text{shock}), 
\end{cases} \quad (74)$$

Once the depth $h_*$ is known the solution for the velocity $u_*$ is

$$u_* = \frac{1}{2}(u_L + u_R) + \frac{1}{2} [f_R(h_*, w_R; h_R) - f_L(h_*, w_L; h_L)] \ . \quad (75)$$
Sketch of the Proof: First note that the particle velocity $u_*$ and depth $h_*$ are constant across the contact discontinuity according to (70). In fact $u_*$ and $h_*$ are constant in the entire Star region. Then, the function $f_L$ is used to relate $u_*$ to the left initial condition $Q_L$ across the left wave. In case the left wave is a shock we have the relation (66) and if it is a rarefaction we use (35). Analogously, the function $f_R$ is used to relate $u_*$ to the right initial condition $Q_R$ across the right wave. If the right wave is a shock we have the relation (58) and if it is a rarefaction we use (42). As $u_* = u_* L = u_* R$, see (70), we can eliminate $u_*$ resulting in equation (72).

Then the particle velocity could be written in terms of the function $f_L$, for both the shock and rarefaction cases. See (66) and (35). So we could compute $u_*$ directly from $f_L$ once $h_*$ is known. Alternatively, we could compute $u_*$ directly from $f_R$ using (58) or (42). Solution(75) results from a mean of the two possible solutions. This concludes the proof.
Iterative solution for $h_*$: We need to solve the algebraic non-linear equation (72) for the unknown $h_*$ in the *Star Region*. To my knowledge, there is no close-form solution available to this equation and therefore we must solve it *numerically* through an iteration procedure. To perform this task there are several methods available, one choice being the Newton-Raphson method

$$h^{(k+1)} = h^{(k)} - \frac{f(h^{(k)})}{f'(h^{(k)})},$$

(76)

for $k = 0, 1, \ldots, K$. Here $f'(h)$ denotes the derivative of $f$ with respect to $h$. The iteration (76) is stopped whenever the change in $h$ is smaller than a prescribed a small positive tolerance $TOL$, that is when

$$\Delta h \equiv \left| \frac{h^{(k+1)} - h^{(l)}}{h^{(k+1)} + h^{(l)}}/2 \right| < TOL .$$

(77)

Usually one takes $TOL = 10^{-6}$. Having formulated and solved numerically the equation for $h_*$, the solution for $u_*$ follows directly from (75).
Two-Rarefaction Case and Guess Value

The iterative procedure (76) requires a guess value \( h^{(0)} \) to start the iteration. To this end we use a two-rarefaction type approximation, as we now describe. Assume a-priori that the two non-linear waves associated with the eigenvalues \( \lambda_1 \) and \( \lambda_3 \) are both rarefaction waves. See Fig. 9. Then the functions \( f_L \) and \( f_R \) in (73), (74) respectively are those corresponding to rarefaction waves. Then (72) becomes

\[
f(h) \equiv 2(a - a_L) + 2(a - a_R) + u_R - u_L = 0, \quad (78)
\]

which has exact solution, called the **Two-Rarefaction Solution**

\[
a_{TR} = \frac{1}{2}(a_L + q_R) - \frac{1}{4}(u_R - u_L). \quad (79)
\]

We use

\[
h^{(0)} = \frac{a_{TR}^2}{g}
\]

as a starting value in the iteration procedure (76).
Now we put together all the components of the solution so as to be able to compute the solution $Q(x, t)$ for any given point $(x, t)$ in the $x$-$t$ half plane, $-\infty < x < \infty$ and $t \geq 0$.

We call this task the solution sampling procedure and assume that the depth $h_*$ and velocity $u_*$ in the *Star Region* are known.

![Diagram](image)

**Fig. 10.** Sampling the solution through the complete wave structure, at time $\hat{t}$.
The solution $Q(x, t)$ is sought at a specified time $\hat{t}$ for any $x$ in a finite interval $[x_l, x_r]$ containing the full wave system, as depicted in Fig. 10. Then $Q(x, \hat{t})$ is a function of $x$ alone and gives a profile at time $\hat{t}$. To sample the solution we make use of the contact discontinuity to divide the full domain into the two subregions

$$R_L = \left\{(x, t)/ \frac{x}{t} \leq u_*\right\}, \quad R_R = \left\{(x, t)/ u_* < \frac{x}{t}\right\}. \quad (80)$$

To perform the sampling we represent the solution in terms of the vector of physical variables $W = [h, u, \psi]^T$ and make use of the similarity variable

$$\xi = \frac{x}{\hat{t}} \quad (81)$$

to locate the sampling point and assign the corresponding solution $W(\xi)$. Note that $\xi$ has dimensions of velocity. There are two cases.
• **Sampling point lies to the left of the contact.** The solution \( W(\xi) \) for \( (x, \hat{t}) \in \mathcal{R}_L \) depends on the wave type. There are two possibilities:

*Left shock.* If the left wave is a shock of speed \( S_L \), then \( \mathcal{R}_L \) is again subdivided into two subregions and the solution is

\[
W(\xi) \equiv \begin{cases} 
W_{*L} = [h_*, u_*, \psi_L]^T & \text{if } S_L \leq \xi \leq u_* \\
W_L = [h_L, u_L, \psi_L]^T & \text{if } \xi < S_L,
\end{cases}
\]  
(82)

where the shock speed \( S_L \) is given by (68).

*Left rarefaction.* If the left wave is a rarefaction then \( \mathcal{R}_L \) is subdivided into three subregions and the solution is

\[
W(\xi) = \begin{cases} 
W_L = [h_L, u_L, \psi_L]^T & \text{if } \xi \leq u_L - a_L \\
W_{Lfan} = [\hat{h}_L, \hat{u}_L, \psi_L]^T & \text{if } u_L - a_L \leq \xi \leq u_* - a_* \\
W_{*L} = [h_*, u_*, \psi_L]^T & \text{if } u_* - a_* \leq \xi \leq u_*
\end{cases}
\]  
(83)

where \( \hat{h}_L \) and \( \hat{u}_L \) inside the left rarefaction are given by (38).
Sampling point lies to the right of the contact. The solution \( W(\xi) \) for \((x, \hat{t}) \in \mathcal{R}_R\) depends on the type of the left wave present. Again there are two possibilities.

Right shock. If the right wave is a shock of speed \( S_R \), then \( \mathcal{R}_R \) is again subdivided into two subregions and the solution is

\[
W(\xi) \equiv \begin{cases} 
W_{*R} &= [h_*, u_*, \psi_R]^T \quad \text{if} \quad u_* \leq \xi \leq S_R, \\
W_R &= [h_R, u_R, \psi_R]^T \quad \text{if} \quad \xi > S_R, 
\end{cases}
\]  
(84)

where the shock speed \( S_R \) is given by (60).

Right rarefaction. If the right wave is a rarefaction then \( \mathcal{R}_R \) is subdivided into three subregions and the solution is

\[
W(\xi) = \begin{cases} 
W_R &= [h_R, u_R, \psi_R]^T \quad \text{if} \quad \xi > u_R + a_R, \\
W_{Rfan} &= \left[ \hat{h}_R, \hat{u}_R, \psi_R \right]^T \quad \text{if} \quad u_{*R} + a_* \leq \xi \leq u_R + a_R, \\
W_{*R} &= [h_*, u_*, \psi_R]^T \quad \text{if} \quad u_* \leq \xi \leq u_{*R} + a_*, 
\end{cases}
\]  
(85)

where \( \hat{h}_R \) and \( \hat{u}_R \) inside the right rarefaction are given by (43).
Test Problems.
Here we solve two specific Riemann problems for the shallow water equations. Table 1 gives the initial conditions and computational details. Column 2 gives the position of the initial discontinuity and column 3 gives the output time. The remaining columns give the initial conditions for depth \( h \) and velocity \( u \). Figures 11 and 12 show profiles for tests 1 and 2 respectively.

<table>
<thead>
<tr>
<th>Test</th>
<th>( x_0 )</th>
<th>( T_{out} )</th>
<th>( h_L )</th>
<th>( u_L )</th>
<th>( h_R )</th>
<th>( u_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.0</td>
<td>7.0</td>
<td>1.0</td>
<td>2.5</td>
<td>0.1</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>25.0</td>
<td>2.5</td>
<td>1.0</td>
<td>10.0</td>
<td>1.0</td>
<td>-10.0</td>
</tr>
</tbody>
</table>

Table 1 Initial conditions for four Riemann problems for the augmented shallow water equations.
Fig. 11. Test 1. Solution profiles for $h$ and $u$ at the output time $T_{out} = 7.0s$. 
Fig. 12. Test 2. Solution profiles for $h$ and $u$ at the output time $T_{out} = 2.5s$. 
We have studied the augmented one-dimensional shallow water equations and solved exactly the associated Riemann problem.

For further reading I recommend, amongst others, the following books:

if $\xi < u_*$ then (left of contact)
  if $h_* \leq h_L$ then (left rarefaction)
    if $\xi < u_L - a_L$ then (left state)
      $W = W_L$
    end
  end
  if $u_L - a_L \leq \xi \leq u_* - a_*L$ then
    $W = W_{L}^{faan}$ (inside left rarefaction)
  end
  if $u_*L - a_*L \leq \xi$ then
    $W = W_{L}^{star}$ (star-left state)
  end
else (left shock)
  if $\xi < S_L$ then (left state)
    $W = W_L$
  else
    $W = W_{L}^{star}$ (star-left state)
  end
end
else (right of contact)
  if $h_* \leq h_R$ then (right rarefaction)
    if $\xi < u_*R + a_*R$ then (right-star state)
      $W = W_{R}^{star}$
    end
  end
  if $u_*R + a \leq \xi \leq u_R + a$ then (inside right rarefaction)
    $W = W_{R}^{faan}$
  end
  if $u_R + a \leq \xi$ then (right state)
    $W = W_R$
  end
else (right shock)
  if $\xi < S_R$ then (star-right state)
    $W = W_{R}^{star}$
  else
    $W = W_R$ (right state)
  end
end
EXERCISES
Problem 1: eigenstructure in primitive variables.

1. Write the augmented shallow water equations in terms of the vector of primitive variables (or physical variables) \( \mathbf{W} = [h, u, \psi]^T \).
2. Express the above equations in the form
   \[ \partial_t \mathbf{W} + \mathbf{M}(\mathbf{W}) \partial_x \mathbf{W} = 0 \, . \] (87)
3. Find the eigenvalues of the coefficient matrix \( \mathbf{M} \).
4. Find the right eigenvectors \( \mathbf{R}_i \) of \( \mathbf{M} \), with scaling factors \( \alpha_i \).
5. Find the left eigenvectors \( \mathbf{L}_j \) of \( \mathbf{M} \), with scaling factors \( \beta_i \).
6. Find the correct scaling factors so that the left and right eigenvectors are orthonormal.

Problem 2: shock jump conditions.

1. Assume a shock wave of speed \( S \) associated with the \( \lambda_3 = u + a \)-characteristic field, connecting the constant states \( \mathbf{Q}_R \) (ahead) and \( \mathbf{Q}_* \) (behind). Define Mach numbers as follows:
   \[ \mathcal{M}_S = \frac{S}{a} , \quad \mathcal{M}_R = \frac{u_R}{a} \, . \] (88)
Find an explicit expression for $h_*$ in terms of the right state and the Mach numbers.

Find an explicit expression for $u_*$ in terms of the right state and the Mach numbers.

What is $\psi_*$ behind the shock?

**Problem 3: shocks for non-conservative variables.**

Consider the shallow water equations of gas dynamics for mass and momentum. Show that assuming solutions are smooth one can derive a mathematically conservative form in terms of the vector of non-conservative variables $W = [h, u]$ as follows

\[
\partial_t W + \partial_x G(W) = 0, \\
W = \begin{bmatrix} h \\ u \end{bmatrix}, \quad G(W) = \begin{bmatrix} hu \\ u^2 + a^2 \ln h \end{bmatrix}.
\]

Derive the shock jump conditions and obtain an expression for the shock speed.
Compare the shock speed obtained above with that obtained using the conventional shock jump conditions in terms of the vector of physically conserved variables.

**Problem 4: weakly hyperbolic system.** Consider the shallow water equations for mass and momentum with sound speed $a = 0$.

1. Find the eigenvalues. Are they real?
2. Find the right eigenvectors. Are they all linearly independent?

**Problem 5: exact solution of Riemann problem.** Consider the function

$$f(h) \equiv f_L(h, w_L; h_L) + f_R(h, w_R; h_R) + \Delta u = 0, \quad \Delta u \equiv u_R - u_L,$$

first given by (72) for the solution $h_*$ of the Riemann problem in the Star Region.

1. Calculate the first and second derivatives of $f(h)$ with respect to $h$.
2. Sketch curves $f(h)$ against $h$ for three values of $\Delta u \equiv u_R - u_L$ and discuss the behaviour of $f(h)$ in terms of roots of $f(h) = 0$.
3. Find constraints on the initial conditions that guarantee the existence of a unique solution $h_* \geq 0$. What is the role of $\Delta u \equiv u_R - u_L$?
**Problem 6:** *extended shallow water system.* Consider the shallow water equations for mass and momentum with sound speed \( a > \), augmented with two advection equations for two passive scalars \( \psi_1 \) and \( \psi_2 \).

1. Find the eigenvalues. Are they real and distinct?
2. Find the right eigenvectors. Are they all linearly independent?
3. Discuss the structure of the solution of the Riemann problem.

**Problem 7.** Write a program to compute the exact solution of the Riemann Problem for the augmented shallow water equations. Test your code on suitably chosen test problems.