Models and methods for blood flow. 7th April 2014.
Time allowed: 2 hours
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Problem 1. Consider the following initial-value problem for the Burgers equation

$$\left.\begin{array}{ll} \text{PDE:} & \partial_t q + \partial_x f(q) = 0\;, \;\; f(q) = \frac{1}{2}q^2\;, \\ \\ \text{IC:} & q(x,0) = h(x) = \left\{\begin{array}{rl} -1 & \text{if} \;\; x < 0\;, \\ \\ 1 & \text{if} \;\; x > 0\;. \end{array}\right. \end{array}\right\} \quad (1)$$

1. Draw the picture of characteristics associated with the initial condition in the x-t plane.
2. Find the entropy-satisfying solution $q(x,t)$ and draw the corresponding picture of characteristics in the x-t plane.
3. Verify that the above solution satisfies the integral form of the conservation law for an arbitrary quadrilateral control volume.
4. Draw solution profiles at times $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$.

Problem 2. Consider the Riemann problem for the Burgers equation

$$\left.\begin{array}{ll} \text{PDE:} & \partial_t q + \partial_x f(q) = 0\;, \;\; f(q) = \frac{1}{2}q^2\;, \\ \\ \text{IC:} & q(x,0) = h(x) = \left\{\begin{array}{rl} \hat{q} & \text{if} \;\; x < 0\;, \\ \\ -\hat{q} & \text{if} \;\; x > 0\;, \end{array}\right. \end{array}\right\} \quad (2)$$

where $\hat{q} > 0$.

1. Show that the function
$$q(x,t) = \left\{\begin{array}{rl} \hat{q} & \text{if} \;\; x < 0\;, \\ \\ -\hat{q} & \text{if} \;\; x > 0 \end{array}\right. \quad (3)$$
is a *weak solution* of the problem. Hint: select an arbitrary control volume in the x-t plane and show that the integral form of the conservation law is satisfied.
2. Verify that the shock solution has speed zero (stationary shock) by applying the Rankine-Hugoniot condition.
3. Plot the solution on the x-t plane, including the picture of characteristics on either side of the discontinuity.
4. Verify that the solution satisfies the Lax entropy condition.

Problem 3: Godunov's method. Consider the function $q_i(x,t)$ defined defined in the strip $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_n, \infty)$ as

- (a) For $\lambda > 0$
$$q_i(x,t) = \left\{\begin{array}{lll} q_{i-1}^n & \text{if} & \dfrac{x - x_{i-\frac{1}{2}}}{\Delta t} < \lambda\;, \\ \\ q_i^n & \text{if} & \dfrac{x - x_{i-\frac{1}{2}}}{\Delta t} > \lambda \end{array}\right. \quad (4)$$

- (b) For $\lambda < 0$

$$q_i(x,t) = \begin{cases} q_i^n & \text{if} \quad \dfrac{x - x_{i+\frac{1}{2}}}{\Delta t} < \lambda \, , \\[2ex] q_{i+1}^n & \text{if} \quad \dfrac{x - x_{i+\frac{1}{2}}}{\Delta t} > \lambda \, . \end{cases} \tag{5}$$

Select a time Δt such that $\lambda \Delta t < \Delta x$ and calculate the integral

$$\bar{q} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q_i(x, t_n + \Delta t) dx \, . \tag{6}$$

The result of this integral should be Godunov upwind method for case (a) $\lambda > 0$ and for case (b) $\lambda < 0$.

Illustrate your answers by drawing the geometric situation in the x-t plane.

Warning:
Each step in your answers must be rigorously justified using the theory.