Metodi Numerici per l'Ambiente. 20th January 2015. Tempo:2:30 Ore
Prof. Dr Eleuterio Toro

Question 1. Consider the linear advection equation

$$\partial_t q + \lambda \partial_x q = 0 \ , \quad \lambda\text{: constant wave propagation speed} \tag{1}$$

solved with a conservative method

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}) \ , \quad f_{i+\frac{1}{2}}\text{: numerical flux.} \tag{2}$$

Suppose the numerical flux is defined as a weighted average as follows

$$f_{i+\frac{1}{2}} = \alpha(\lambda q_i^n) + (1-\alpha)(\lambda q_{i+1}^n) \ , \quad \alpha\text{: a real number (a parameter).} \tag{3}$$

1. Write the resulting scheme as the three-point scheme
$$q_i^{n+1} = b_{-1} q_{i-1}^n + b_0 q_i^n + b_1 q_{i+1}^n \ , \tag{4}$$
identifying the expressions for the coefficients.
2. Assume $\lambda > 0$ and find the range of values of α that produce a monotone method.
3. Find the value of α (in terms of other parameters) so that the scheme (4) is second-order accurate in space and time. Enforce second-order accuracy by applying the accuracy lemma.

Question 2. Consider the linearised shallow water equations

$$\left.\begin{array}{r} \partial_t \eta + H \partial_x v = 0 \ , \\ \partial_t v + g \partial_x \eta = 0 \ , \end{array}\right\} \tag{5}$$

where H is the, constant, unperturbed water depth and g is the acceleration due to gravity. The unknowns functions of the problem are $\eta(x,t)$: water depth and $v(x,t)$: particle velocity. Perform the following tasks:

1. Express equations (5) as a system in matrix form
$$\partial_t \mathbf{Q} + \mathbf{M} \partial_x \mathbf{Q} = \mathbf{0} \ , \tag{6}$$
clearly identifying $\mathbf{Q}$ and $\mathbf{M}$.
2. Compute the eigenvalues and corresponding left and right eigenvectors of (6).
3. Apply the orthonormality condition of left and right eigenvectors to calculate the otherwise arbitrary scaling factors for the eigenvectors.
4. Express equations (6) in terms of characteristic variables.

Question 3. Consider the following initial-value problem for the traffic flow equation

$$\left.\begin{array}{ll} \text{PDE:} & \partial_t q + \partial_x f(q) = 0 \ , \ \ f(q) = f(q) = u_{max}(1 - q/q_{max})q \ , \\ \text{IC:} & q(x,0) = h(x) = \left\{ \begin{array}{ll} q_L & \text{if} \quad x < 0 \ , \\ q_R & \text{if} \quad x > 0 \ . \end{array} \right. \end{array}\right\} \tag{7}$$

1. Find the characteristic speed $\lambda(q)$.
2. Assume the initial data q_L, q_R to be connected by a shock wave. Apply the Rankine-Hugoniot conditions and find an expression for the shock speed.
3. Find conditions on the initial data q_L, q_R for the shock to be entropy satisfying. Compare your conclusions with the Burgers equation case.
4. What are the conditions on the initial data q_L, q_R so as to generate a rarefation (entropy satisfying) solution.