COMPUTATIONAL HAEMODYNAMICS.
10th June 2015. Time allowed: 2:30 hours
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Question 1. Consider the linear advection equation

$$\partial_t q + \lambda \partial_x q = 0 \ , \quad \lambda\text{: constant wave propagation speed} \tag{1}$$

solved with a conservative method

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}) \ , \quad f_{i+\frac{1}{2}}\text{: numerical flux.} \tag{2}$$

Suppose the numerical flux is defined as a weighted average of the fluxes to the left and right of the interface, as follows

$$f_{i+\frac{1}{2}} = \alpha(\lambda q_i^n) + (1-\alpha)(\lambda q_{i+1}^n) \ , \quad \alpha\text{: a real number, a parameter.} \tag{3}$$

1. Write the resulting scheme as the three-point scheme

   $$q_i^{n+1} = b_{-1} q_{i-1}^n + b_0 q_i^n + b_1 q_{i+1}^n \ , \tag{4}$$

   identifying the expressions for the coefficients.
2. Considering, separately, both cases $\lambda > 0$ and $\lambda < 0$, find the range of values of α that produce a monotone method.
3. Comment on your findings, relating them to standard schemes, such as the Godunov method and others.

Question 2. Consider the linearised equations for blood flow

$$\partial_t a + A_0 \partial_x v = 0 \ , \qquad \partial_t v + \frac{c_0^2}{A_0} \partial_x a = 0 \ , \tag{5}$$

where the unknowns are: $a(x,t)$ and $v(x,t)$, respectively for cross-sectional area and velocity. A_0 is a constant representing the equilibrium cross-sectional area and c_0 is the *wave speed* given as

$$c_0 = \sqrt{\frac{\beta\sqrt{A_0}}{2\rho}} \ , \tag{6}$$

where β is a positive constant parameter representing properties of the blood vessels.

Perform the following tasks:

1. Express equations (5) as a system in matrix form

   $$\partial_t \mathbf{Q} + \mathbf{M} \partial_x \mathbf{Q} = \mathbf{0} \ , \tag{7}$$

   clearly identifying $\mathbf{Q}$ and $\mathbf{M}$.
2. Compute the eigenvalues and left and right eigenvectors of (7).
3. Apply the orthonormality condition of left and right eigenvectors to calculate the otherwise arbitrary scaling factors for the eigenvectors.
4. Express equations (7) in terms of characteristic variables.

5. Solve the Riemann problem for (7), written as

$$\left.\begin{array}{ll} \text{PDEs:} & \partial_t \mathbf{Q} + \mathbf{M}\partial_x \mathbf{Q} = \mathbf{0}\ , \\ \text{ICs:} & \mathbf{Q}(x,0) = \mathbf{Q}^{(0)}(x) = \begin{cases} \mathbf{Q}_L = [a_L, v_L]^T & \text{if} \quad x < 0\ , \\ \mathbf{Q}_R = [a_R, v_R]^T & \text{if} \quad x > 0\ . \end{cases} \end{array}\right\} \tag{8}$$

Question 3. Consider the general balance law

$$\partial_t q + \partial_x f(q) = s(q)\ . \tag{9}$$

1. Integrate (8) on the finite volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_n, t_{n+1}]$ to obtain

$$q_i^{n+1} = q_i^n - \frac{\Delta t}{\Delta x}[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}] + \Delta t s_i\ , \tag{10}$$

   giving the correct definitions for q_i^n, $f_{i+\frac{1}{2}}$ and s_i.

2. Recall that formula (10) may be used to construct finite volume methods. In such case comment on formula (10) and the meaning of q_i^n, $f_{i+\frac{1}{2}}$ and s_i.

Question 4. Construct, step by step, the second-order ADER scheme to solve the linear advection equation with a linear source term

$$\partial_t q + \lambda \partial_x q = \beta q\ . \tag{11}$$

Note: concerning the reconstruction slopes Δ_i, you do not need to make particular choices.

Warning:
Each step in your answers must be rigorously justified using the theory.